CIS 160

Recitation Guide - Week 9

Topics Covered: Expectation, Linearity of Expectation, Spanning Trees

Problem 1:
Taki says to Yuyang, "Let's play a game. I first roll a fair 6-sided die. If the number that shows up is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll. If not, then I flip a fair coin. If it is tails, I take 10 dollars from you, and if it is heads, I pay you 5 dollars. What is your expected payoff?"

Yuyang, who does not want to lose to Taki again, has asked you to help him out. Should he play the game?

Solution:

Define our sample space to be all the ways for the game to play out. (Formally, we can say $\Omega = \{(d_1, d_2) | d_1 \in \{3, 6\}, d_2 \in \{1, 2, 3, 4, 5, 6\}\} \cup \{(d, f) | d \in \{1, 2, 4, 5\}, f \in \{H, T\}\}$.)

We will denote the payoff using a random variable X. In this case, it is actually easiest to determine $\mathbf{E}[X]$ using the basic formula for expected value, where we calculate the probability and payoff of each outcome in the sample space.

We assume independence between die rolls and coin flips. Since we have a fair die and a fair coin, we have a uniform probability distribution within each die roll and coin flip. For each outcome in which the first roll is divisible by 3 (which happens with probability $\frac{2}{6} = \frac{1}{3}$), then the payoff for the outcome is just equal to the value of the second roll, and the probability of each outcome is $\frac{1}{3} \cdot \frac{1}{6}$.

For each outcome in which the first roll is not divisible by 3, then the payoffs are equal to -10 or 5, depending on the coin flip, and both outcomes have probability $\frac{2}{3} \cdot \frac{1}{2}$. Thus, the expected value is:

$$
\begin{aligned}
\mathbf{E}[X] &= \sum_{\omega \in \Omega} \Pr(\omega) \cdot X(\omega) \\
&= \frac{1}{3} \cdot \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) + \frac{2}{3} \cdot \frac{1}{2}(-10 + 5) \\
&= \frac{1}{18}(21) + \frac{1}{3}(-5) \\
&= \boxed{-\frac{1}{2}}
\end{aligned}
$$

Since the expected payoff is negative, Yuyang should not play the game.

Problem 2:
There are n people in a room. Each pair of people has probability p of being friends (uniform probability across all pairs of people). What is the expected number of friend groups of size m in the room (in terms of n, p, and m)? Friend groups are groups of people in which everyone in the group is friends with everyone else in the group. Note that a person can be in more than one friend group.

Solution:

Let the sample space Ω be the set of all possible friendship assignments to each pair of the n people. Further, let X be the random variable denoting the number of friend groups of size m. Note that there are $\binom{n}{m}$ possible groups of size m. We define an indicator random variable X_i where $X_i = 1$ if group i is a friend group and 0 otherwise. Thus we can express X as follows:

$$X = \sum_{i=1}^{\binom{n}{m}} X_i$$

Assuming independence between pairs, then for all i, $\Pr[X_i = 1]$ is $p^{\binom{m}{2}}$ because every possible friendship between any two people in the group must exist. Also note that because these are indicator random variables, $\mathbf{E}[X_i]$ is also equal to $p^{\binom{m}{2}}$.

By linearity of expectation, we can compute the following:

$$\begin{aligned}
\mathbf{E}[X] &= \sum_{i=1}^{\binom{n}{m}} \mathbf{E}[X_i] \\
&= \sum_{i=1}^{\binom{n}{m}} p^{\binom{m}{2}} \\
&= \boxed{\binom{n}{m} \cdot p^{\binom{m}{2}}}
\end{aligned}$$

Problem 3: Consider a connected graph $G = (V, E)$ and an arbitrary partition of G's vertex set V into nonempty sets S and $V \setminus S$. Prove that if there exists only one edge e between the vertices in S and the vertices in $V \setminus S$, then e must be in every spanning tree of G.

Solution:

Consider an arbitrary spanning tree of G, say T. Since T is a tree, we know that it is connected, and thus there is a path between any pair of vertices.

Consider a vertex $x \in S$ and consider another vertex $y \in V \setminus S$. Because T is connected, there must be a path P from x to y in T. Let us consider this path.

We define edges that cross the cut between S and $V \setminus S$ to have an endpoint in S and an endpoint in $V \setminus S$. We know that P goes from a vertex in S to a vertex in $V \setminus S$. Therefore, there must exist an edge in the path that crosses the cut between S and $V \setminus S$ (if not, then the path would always stay in either S or $V \setminus S$, which it clearly doesn't). However, by our assumption we know that the only edge that crosses the cut is e. Therefore, our path P contains e and hence, our tree T must contain e.