CIS 160 Recitation 9

Linearity of Expectation, Indicators, Variance, Markov's Inequality, Graph Coloring and Matching

November 4-5, 2021

Linearity of Expectation (LOE)

- The expectation of the sum of random variables equals the sum of their expectations.
- For random variables $X_1, X_2, ..., X_n$ on the same probability space Ω and $c_1, c_2, ..., c_n \in \mathbb{R}$

$$E[\sum_{i=1}^{n} c_i X_i] = \sum_{i=1}^{n} c_i E[X_i]$$

- Random variables do not have to be independent.

Indicators

- An indicator I_A of the event A in the sample space Ω is defined by

$$I_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}$$

- We can think of I_A as a Bernoulli r.v. with success probability $Pr[A]$

$$E[I_A] = Pr[I_A = 1] = Pr[A]$$

Variance

- How much a random variable deviates from its mean.
- The variance of a random variable X is defined as

$$Var[X] = E[(X - E[X])^2] = E[X^2] - (E[X])^2$$

- The standard deviation of a random variable X is

$$\sigma[X] = \sqrt{Var[X]}$$

- If X and Y are independent random variables, then $Var[X + Y] = Var[X] + Var[Y]$ and $E[XY] = E[X]E[Y]$

Markov's Inequality

Let X be a non-negative random variable. For all $a > 0$:

- **Markov's Inequality**

$$Pr[X \geq a] \leq \frac{E[X]}{a}$$

Graph coloring

- A graph is *k-colorable* if each vertex can be colored using one of the k colors so that adjacent vertices are colored using different colors.
- The *chromatic number* of a graph G, $\chi(G)$, is the smallest value of k for which G is k-colorable
- A *bipartite* graph is a graph that is 2-colorable.
- A graph with maximum degree at most k is $(k+1)$-colorable.

Matching

- A *matching* in a graph is a set of edges with no shared end-points.
- A *perfect matching* in a graph is a matching that saturates every vertex in the graph.