CIS 160 Recitation #8

Trees, Independence, and Random Variables

Trees

A **tree** is a connected, acyclic graph.

A **forest** is an acyclic graph (each connected component is a tree).

Example. For a n-vertex graph G, the following are equivalent and characterize trees with n vertices.

(1) G is a tree.
(2) G is connected and has exactly $n-1$ edges.
(3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.
(4) G contains no cycle but $G + \{x, y\}$ does, for any two non-adjacent vertices $x, y \in G$.
(5) Any two vertices of G are linked by a unique path in G.

Independence

Two events A and B are independent if and only if $\Pr[A \cap B] = \Pr[A] \cdot \Pr[B]$

If we have events $A_1, A_2, \ldots, A_n$, we say that they are mutually independent if for every subset $I \subseteq \{1, 2, \ldots, n\}$:

$$\Pr[\cap_{i \in I}] = \prod_{i=1} \Pr[A_i]$$

Random Variables

A random variable maps each sample point in the sample space to a real number.

For a discrete random variable X and a real value a, the event $X = a$ is the set of sample points in the sample space that are mapped to a.

$$\Pr[X = a] = \sum_{\omega \in \Omega : X(\omega) = a} \Pr[\omega]$$

The distribution or probability mass function (PMF) gives the probabilities for different possible real values that the random variable can take on. These probabilities must sum to 1.

Independence of Random Variables

Two random variables X and Y are independent if and only if:

$$\forall x, y, \Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \cdot \Pr[Y = y]$$

Random variables $X_1, X_2, \ldots, X_k$ are mutually independent if and only if, for any subset $I \subseteq [1, k]$ and any values $x_i, i \in I$:

$$\Pr[\cap_{i \in I} X_i = x_i] = \prod_{i \in I} \Pr[X_i = x_i]$$