CIS 160

Recitation Guide - Week 8

Topics Covered: Expectation, Independence, Spanning Trees

Problem 1:

Taki says to Yuyang, "Let's play a game. I first roll a fair 6-sided die. If the number that shows up is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll. If not, then I flip a fair coin. If it is tails, I take 10 dollars from you, and if it is heads, I pay you 5 dollars. What is your expected payoff?"

Yuyang has asked you to help him out. Should he play the game?

Solution:

Define our sample space to be all the ways for the game to play out. (Formally, we can say $\Omega = \{(d_1, d_2) \mid d_1 \in \{3, 6\}, d_2 \in \{1, 2, 3, 4, 5, 6\}\} \cup \{(d, f) \mid d \in \{1, 2, 4, 5\}, f \in \{H, T\}\}$.)

We will denote the payoff using a random variable X. In this case, it is actually easiest to determine $\mathbf{E}[X]$ using the basic formula for expected value, where we calculate the probability and payoff of each outcome in the sample space.

We assume independence between die rolls and coin flips. Since we have a fair die and a fair coin, we have a uniform probability distribution within each die roll and coin flip. For each outcome in which the first roll is divisible by 3 (which happens with probability $\frac{2}{6} = \frac{1}{3}$), then the payoff for the outcome is just equal to the value of the second roll, and the probability of each outcome is $\frac{1}{3} \cdot \frac{1}{6}$.

For each outcome in which the first roll is not divisible by 3, then the payoffs are equal to -10 or 5, depending on the coin flip, and both outcomes have probability $\frac{2}{3} \cdot \frac{1}{2}$. Thus, the expected value is:

$$\begin{aligned}
\mathbb{E}[X] &= \sum_{\omega \in \Omega} \Pr(\omega) \cdot X(\omega) \\
&= \frac{1}{3} \cdot \frac{1}{6}(1 + 2 + 3 + 4 + 5 + 6) + \frac{2}{3} \cdot \frac{1}{2}(-10 + 5) \\
&= \frac{1}{18}(21) + \frac{1}{3}(-5) \\
&= \boxed{-\frac{1}{2}}
\end{aligned}$$

Since the expected payoff is negative, Yuyang should not play the game.

Problem 2:
We have three wooden buckets, A, B, C and we throw $n \geq 3$ metal keys in them. The key throws are mutually independent and each key is equally likely to land in each of the three buckets.

(a) Let A be the event that after all keys are thrown, bucket A has at least one key in it and similarly associate an event B with B. Are A and B independent? Justify your answer.

(b) Compute the probability that after all keys are thrown, each of the three buckets has at least one key in it. Justify your answer.

Solution:

Define $\Omega = \{(x_1, x_2, ..., x_n) \mid x_i \in \{A, B, C\}\}$. That is, the sample space is the set of all n-tuples where the ith element in the tuple represents the bucket that key i landed in.

(a) For $i = 1, \ldots, n$ let A_i be the event that key i is thrown in bucket A. We have $\Pr[A_i] = \frac{1}{3}$. Clearly $A = A_1 \cup \cdots \cup A_n$ and since the events $A_1, \ldots, A_n$ are mutually independent we can compute:

$$\Pr[A] = \Pr[A_1 \cup \cdots \cup A_n] \;=\; 1 \;-\; \prod_{i=1}^{n} (1 - \Pr[A_i]) \;=\; 1 - \left(1 - \frac{1}{3}\right)^n \;=\; 1 - \left(\frac{2}{3}\right)^n$$

Similarly, $\Pr[B] = 1 - \left(\frac{2}{3}\right)^n$. To check independence we also need $\Pr[A \cap B]$.

Upon reflection, we notice that there is one aspect of the problem that we have not used yet: the keys get thrown *only* in A, B and C. Thus, $\bar{A} \cap \bar{B}$, which means that both A and B are empty after all keys are thrown, is the same as the event "all keys get thrown in C" and therefore, by mutual independence, has probability $\left(\frac{1}{3}\right)^n$, as each key has a $\frac{1}{3}$ probability of being thrown into C. Now we can compute, using properties of probability and De Morgan's Laws:

$$\begin{aligned}
\Pr[A \cap B] &= \Pr[A] + \Pr[B] - \Pr[A \cup B] \\
&= \Pr[A] + \Pr[B] - \left(1 - \Pr[\overline{A \cup B}]\right) \\
&= \Pr[A] + \Pr[B] - \left(1 - \Pr[\bar{A} \cap \bar{B}]\right) \\
&= 1 - \left(\frac{2}{3}\right)^n + 1 - \left(\frac{2}{3}\right)^n - \left(1 - \left(\frac{1}{3}\right)^n\right) \\
&= 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n
\end{aligned}$$

We also know that:

$$\Pr[A] \cdot \Pr[B] = \left(1 - \left(\frac{2}{3}\right)^n\right)\left(1 - \left(\frac{2}{3}\right)^n\right) = 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

Now we check for independence,

$$\Pr[A \cap B] \stackrel{?}{=} \Pr[A] \cdot \Pr[B]$$
$$1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n \neq 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{4}{9}\right)^n$$

because $\frac{1}{3} \neq \frac{4}{9}$, and so A and B are not independent.

(b) We continue with the notation introduced in part (a) and we also define C to be the event "C is not empty after all keys are thrown. This part asks for $\Pr[A \cap B \cap C]$. We are tempted to multiply probabilities but we do not know if A, B, C are mutually independent. In fact, in part (a) we saw that $A \not\perp B$. Although it is still possible that $\Pr[A \cap B \cap C] = \Pr[A] \cdot \Pr[B] \cdot \Pr[C]$ there is no reason to hope for this here (and in fact we shall see that it does not hold).

Instead, we will use the Principle of Inclusion-Exclusion for three events:

$$\Pr[A \cup B \cup C] = \Pr[A] + \Pr[B] + \Pr[C] - \Pr[A \cap B] - \Pr[B \cap C] - \Pr[C \cap A] + \Pr[A \cap B \cap C]$$

Since we have at least one key, at least one of the buckets ends up non-empty. Hence $A \cup B \cup C = \Omega$, meaning $A \cup B \cup C$ consists of all the outcomes and has probability 1. From part (a) we have:

$$\Pr[A] = \Pr[B] = \Pr[C] = 1 - \left(\frac{2}{3}\right)^n$$
$$\Pr[A \cap B] = \Pr[B \cap C] = \Pr[C \cap A] = 1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n$$

We plug in and obtain

$$\begin{aligned}\Pr[A \cap B \cap C] &= \Pr[A \cup B \cup C] - \Pr[A] - \Pr[B] - \Pr[C] + \Pr[A \cap B] + \Pr[B \cap C] + \Pr[C \cap A] \\ &= 1 - 3\left(1 - \left(\frac{2}{3}\right)^n\right) + 3\left(1 - 2\left(\frac{2}{3}\right)^n + \left(\frac{1}{3}\right)^n\right) \\ &= 1 - 3\left(\frac{2}{3}\right)^n + 3\left(\frac{1}{3}\right)^n\end{aligned}$$

Problem 3: Consider a connected graph $G = (V, E)$ and an arbitrary partition of G's vertex set V into nonempty sets S and $V \setminus S$. Define edges that cross the cut between S and $V \setminus S$ to have an endpoint in S and an endpoint in $V \setminus S$. Prove that if there exists only one edge e that crosses the cut S and $V \setminus S$, then e must be in every spanning tree of G.

Solution:

Consider an arbitrary spanning tree of G, say T. Since T is a tree, we know that it is connected, and thus there is a path between any pair of vertices.

Consider a vertex $x \in S$ and consider another vertex $y \in V \setminus S$. Because T is connected, there must be a path P from x to y in T. Let us consider this path. We will prove that e must be on path P.

We can prove this as follows. Assume for contradiction that e is not on path P. Consider the first occurrence of a vertex $u \in V \setminus S$ on path P. Since P starts with a vertex in S, and $u \in V \setminus S$, u must not be the first vertex on the path P. Hence, there exists a vertex $w \in S$ on the path P directly preceding u, so we have the edge $\{w, u\}$ on path P. However, this edge is an edge between a vertex in S and a vertex in $V \setminus S$ that is not edge e, contradicting e being the only edge between these two sets of vertices.

Therefore, our path P contains e and hence, our tree T must contain e.