CIS 160 Recitation 8

Independence, Random Variables, Expectation, Eulerian & Hamiltonian Graphs

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Independence

- Two events A and B are independent iff

$$Pr[A \cap B] = Pr[A] \times Pr[B]$$

- If two events A and B are independent and $\Pr[B] > 0$, then

$$Pr[A|B] = Pr[A]$$

Pairwise Independence

- Events $A_1, A_2, \cdots, A_n$ are **pairwise** independent if for all $i, j \in [1..n]$,
$$Pr[A_i \cap A_j] = Pr[A_i] \cdot Pr[A_j]$$

Mutual Independence

- Events $A_1, A_2, \cdots, A_n$ are **mutually** independent if for any $\{i_1, ..., i_k\} \subseteq [1..n]$,

$$Pr[A_{i_1} \cap ... \cap A_{i_k}] = Pr[A_{i_1}] \cdots Pr[A_{i_k}]$$

- Note that $Pr[A_1 \cap \cdots \cap A_n] = Pr[A_1] \cdots Pr[A_n]$ is not a sufficient condition for $A_1, A_2, ..., A_n$ to be mutually independent.
- Mutual independence implies pairwise independence but the converse is **not** true.

Random Variables

- A random variable X on Ω is a function that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.
- $X = a$ is the set of outcomes in Ω for which the r.v. takes the value a.

$$Pr[X = a] = \sum_{\omega \in \Omega : X(\omega) = a} Pr[\omega]$$

Random Variables

- Note that $\sum_x Pr[X = x] = 1$ since events $X = x$ are disjoint and partition Ω.
- Random variables $X_1, X_2, ..., X_n$ are mutually independent if for any subset $I \subseteq [1..n]$ and any values x_i, where $i \in I$,

$$Pr[\cap_{i \in I}(X_i = x_i)] = \prod_{i \in I} Pr[X_i = x_i]$$

Expectation

- The weighted average (proportional to the probabilities) of the possible values of X.
- $\mathbb{E}[X]$ is the value we would expect to obtain if we repeated a random experiment many times and took the average of the outcomes of X.

$$\mathbb{E}[X] = \sum_i i \cdot Pr[X = i]$$

Eulerian & Hamiltonian Graphs

- An Eulerian circuit is a closed walk in which each edge appears exactly once.
- A connected graph is Eulerian if it contains an Eulerian circuit.
- A connected graph G is Eulerian iff every vertex in G has even degree.
- A Hamiltonian cycle in a graph G is a cycle in which each vertex of G appears exactly once.
- A graph is Hamiltonian if it contains a Hamiltonian cycle.