Topics Covered: Expectation, Independence, Spanning Trees

Problem 1:
Taki says to Yuyang, “Let’s play a game. I first roll a fair 6-sided die. If the number that shows up is divisible by 3, I roll again and I pay you the dollar amount that shows up on the second roll. If not, then I flip a fair coin. If it is tails, I take 10 dollars from you, and if it is heads, I pay you 5 dollars. What is your expected payoff?”

Yuyang has asked you to help him out. Should he play the game?
Problem 2:

We have three wooden buckets, $A, B, C$ and we throw $n \geq 3$ metal keys in them. The key throws are mutually independent and each key is equally likely to land in each of the three buckets.

(a) Let $A$ be the event that after all keys are thrown, bucket $A$ has at least one key in it and similarly associate an event $B$ with $B$. Are $A$ and $B$ independent? Justify your answer.

(b) Compute the probability that after all keys are thrown, each of the three buckets has at least one key in it. Justify your answer.
Problem 3: Consider a connected graph $G = (V, E)$ and an arbitrary partition of $G$'s vertex set $V$ into nonempty sets $S$ and $V \setminus S$. Define edges that cross the cut between $S$ and $V \setminus S$ to have an endpoint in $S$ and an endpoint in $V \setminus S$. Prove that if there exists only one edge $e$ that crosses the cut $S$ and $V \setminus S$, then $e$ must be in every spanning tree of $G$. 