CIS 160

Recitation Guide - Week 7

Topics Covered: Graphs, Trees, Conditional Probability

Problem 1:
Let T be a tree where the maximum degree is Δ. Prove that T has at least Δ leaves.

Solution:

We will use the (non-standard) notation $\lambda(T)$ to denote the number of leaves in a tree T. Thus, we can rewrite the claim as $\lambda(T) \geq \Delta$.

We first prove a lemma.

Lemma: All maximal paths in a tree must start and end with leaves.

Proof: Suppose for the sake of contradiction that at least one of the endpoints are not leaves. Let v be this vertex. Notice that $deg(v) \geq 2$, since it is not a leaf. As it is part of a maximal path, there must exist some vertex u that is a neighbor of v, but not the neighbor of v in the maximal path. However, this creates a cycle, as u must lie on the maximal path (or the path could be extended), which is a contradiction since trees do not have cycles. ■

Let $v \in V$ have degree Δ. For each $u_i, u_j \in N(v)$, let $P_{i,j}$ be a maximal path including $u_i - v - u_j$. Note that there must be at least $\binom{\Delta}{2}$ such paths, such that each pair of starting edges gives a different path. We know from the above lemma that any such path $P_{i,j}$ must terminate in two leaves. Lastly, note that since there is a unique path between any two vertices in a tree, every pair of leaves admits at most one maximal path. Therefore, if there were $\lambda(T) < \Delta$ leaves, we would end up with $\binom{\lambda}{2} < \binom{\Delta}{2}$ distinct maximal paths, a contradiction. We must then have $\lambda(T) \geq \Delta$.

Problem 2:

You run into a town with 100 robots. You know that 99 of these robots tell the truth half the time and lie the other half. You also know that there is exactly 1 truthful robot in town, who always tells the truth. You take a robot at random and ask a question seven times, and the robot tells the truth every time. What is the probability that this is the truthful robot?

Solution:

We can define each outcome in the sample space Ω as an ordered pair, where the first element represents whether or not the robot is truthful, and the second element is the robot's answers to the 7 questions, i.e. a sequence of length 7 of elements from $\{T,F\}$.

Let A be the event: selected robot is truthful. Let B be the event: selected robot tells the truth seven times.

Note that from parsing the question, we know the following probabilities:

$$\Pr[A] = \frac{1}{100} \qquad \Pr[\overline{A}] = \frac{99}{100}$$

$$\Pr[B|A] = 1 \qquad \Pr[B|\overline{A}] = \left(\frac{1}{2}\right)^7$$

We want $\Pr[A|B]$ (make sure you can justify each step in this simplification):

$$\begin{aligned}
\Pr[A|B] &= \frac{\Pr[A \cap B]}{\Pr[B]} \\
&= \frac{\Pr[A] \times \Pr[B|A]}{\Pr[B]} \\
&= \frac{\Pr[A] \times \Pr[B|A]}{\Pr[A \cap B] + \Pr[\overline{A} \cap B]} && \text{(Total Probability Theorem)} \\
&= \frac{\Pr[A] \times \Pr[B|A]}{\Pr[A] \times \Pr[B|A] + \Pr[\overline{A}] \times \Pr[B|\overline{A}]} \\
&= \frac{\frac{1}{100} \times 1}{\frac{1}{100} \times 1 + \frac{99}{100} \times (\frac{1}{2})^7} \\
&= \frac{128}{227} \approx \boxed{0.564}
\end{aligned}$$

Problem 3:
Prove that G or the complement of G is connected. Note that the complement of a graph $G = (V, E)$ is $G' = (V', E')$ such that $V = V'$ and $\forall u, v \in V$, $(u, v) \in E' \iff (u, v) \notin E$.

Solution:

If G is connected we are done.

If G is not connected then G is composed of multiple connected components. We want to prove that given two arbitrary vertices in G there must be a path between them in $\overline{G}$. Let these two arbitrary vertices be u and v.

Case 1: u and v do not share an edge in G

This means they must share an edge in $\overline{G}$ and thus there is a path from u to v in $\overline{G}$.

Case 2: u and v share an edge in G

This means they were part of the same connected component in G. Take an arbitrary vertex x in a different connected component in G. Edges $u - x$ and $v - x$ must both exist in $\overline{G}$. Thus, there is a path $u - x - v$ between vertices u and v.

Thus, we have shown that there exists a path between any two arbitrary vertices in $\overline{G}$. By definition $\overline{G}$ must be connected. The claim is proved.