

Recitation Guide - Week 6

Topics Covered: Graphs, Probability

Problem 1:

In this problem we illustrate a common trap that we can fall in when proving statements about graphs by induction on the number of vertices or the number of edges. Here is a *false statement*: “If every vertex in a simple graph G has strictly positive (> 0) degree, then G is connected”.

- (a) Prove that the statement is indeed false by providing a counterexample.
- (b) Since the statement is false, there must be something wrong in the following “proof”. Pinpoint the *first* logical mistake (unjustified step).

Buggy Proof:

We prove the statement by induction on the number of vertices. Let $P(n)$ be the following proposition: “for any graph with n vertices, if every vertex has strictly positive degree, then the graph is connected”.

Base Cases: Notice that $P(1)$ is vacuously true. We also show that $P(2)$ is true. Notice that there is only one graph with two vertices of strictly positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Induction Hypothesis: Assume that for some $k \geq 2$, $P(k)$ is true.

Induction Step:

Consider a graph G_{old} with k vertices in which every vertex has strictly positive degree. By the Induction Hypothesis this graph is connected. Now we add one more vertex, call it u , to obtain a graph G_{new} with $k + 1$ vertices.

All that remains is to check that in G_{new} there is a walk from u to every other vertex v . Since u has positive degree, there is an edge from u to some other vertex, say w . But w and v are in G_{old} , which is connected, and therefore there is a walk from w to v . This gives a walk $u - w - v$ in G_{new} . ✓

- (c) Now consider the changed Induction Step and identify a mistake in this proof.

Induction Step:

Consider a graph G with $k + 1$ vertices in which every vertex has strictly positive degree. Remove an arbitrary vertex, call it u , and now we have a graph G' with k vertices. By the Induction Hypothesis this graph is connected. Now we add u back in to obtain a graph G with $k + 1$ vertices.

All that remains is to check that in G there is a walk from u to every other vertex v . Since u has positive degree, there is an edge from u to some other vertex, say w . But w and v are in G' , which is connected, and therefore there is a walk from w to v . This gives a walk $u - w - v$ in G . ✓

Solution:

- (a) Consider the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{(a, b), (c, d)\}$. Every vertex has degree one, however the graph is not connected (there is no path from a to c , for example).
- (b) The logical mistake in the proof is where we “add one more vertex” in the induction step. It is certainly possible to add one more vertex to a graph such that all vertices have strictly positive degree, but this constructs a *particular* type of graph G_{new} with $k + 1$ vertices, where we actually had to show $P(k + 1)$, which is that the claim holds for any graph with $k + 1$ vertices. In particular, there are graphs with $k + 1$ vertices where all its vertices have strictly positive degree that cannot be constructed from graphs with k vertices that fulfill the same condition. For instance, there does not exist any graph with 3 vertices where all its vertices have strictly positive degree such that by adding a new vertex we obtain graph G in part a). This highlights the importance of starting with an arbitrary graph with $k + 1$ vertices, then deconstruct it to obtain a graph with k vertices to apply the IH to in graph induction proofs!

There are a couple of statements that may seem “bogus” but are actually not. They are as follows.

- (a) “ $P(1)$ is vacuously true”: This is not “bogus”, as a simple graph with 1 vertex must not have any edges, so it cannot have strictly positive degree.
- (b) “Let k be an arbitrary integer such that $k \geq 2$ ”: This is not “bogus”, as we have an additional base case for $n = 2$, while $P(1)$ is proved separately.
- (c) After removing a vertex, we have to make sure that in G' , the properties specified in IH still exist. In this case, we have to make sure that after removing a vertex, every vertex still has a strictly positive degree to apply IH.

Consider the neighbors of u in G . If there was a neighbor x such that the degree of x in G was 1, since its only neighbor was removed, its degree in G' would be 0. Therefore, we cannot always apply IH to G' .

Problem 2:

Compute the probability of the event “when we roll two identical 6-sided beige dice the numbers add up to an even number.”

Solution:

We first describe the sample space for this problem by:

$$\Omega = \{\{x, y\} \mid x, y \in [1..6]\} = \{\{x, y\} \mid x, y, x \neq y\} \cup \{x-x \mid x \in [1..6]\}$$

Note that our probability distribution is not uniform.

Let E be the event where the sum of the two rolls results in an even number. Note that we have $\binom{6}{2} = 15$ outcomes in which the dice show different numbers; each of these has probability $2 \times \frac{1}{36} = \frac{1}{18}$. Among these outcomes, the numbers add up to an even number if they are both odd, and there are 3 of these, $\{1, 3\}, \{1, 5\}, \{3, 5\}$, or if they are both even – there also 3 of these: $\{2, 4\}, \{2, 6\}, \{4, 6\}$. So that’s 6 outcomes of probability $\frac{1}{18}$ each in which the numbers are different.

We also have 6 more outcomes in which the die show the same number; each of these has probability $\frac{1}{36}$. In all these outcomes the numbers add up to an even number, hence we have another 6 outcomes of probability $\frac{1}{36}$ each.

We now calculate the desired probability using the definition of event:

$$\begin{aligned} \Pr[E] &= \sum_{w \in E} \Pr(w) \\ &= \Pr[\{1, 3\}] + \Pr[\{1, 5\}] + \Pr[\{3, 5\}] + \Pr[\{2, 4\}] + \Pr[\{2, 6\}] + \Pr[\{4, 6\}] + \sum_{x \in [1..6]} \Pr[x-x] \\ &= 6 \times \frac{1}{18} + 6 \times \frac{1}{36} \\ &= \frac{1}{3} + \frac{1}{6} \\ &= \frac{1}{2} \end{aligned}$$

Note that since “adding up to even” is an event in which the die color doesn’t matter, we could have provided an equivalent solution that assumes the dice are green-purple rather than beige-beige. Doing so would change the sample space, which leads to the following alternate solution.

Alternate Solution:

We first define the sample space for this problem as:

$$\Omega = \{(x, y) \mid x, y \in [1..6]\} = \{(x, y) \mid x, y \in [1..6], x \neq y\} \cup \{(x, x) \mid x \in [1..6]\}$$

This sample space gives us a uniform probability distribution, where each outcome has probability $\frac{1}{36}$.

Let E be the event where the sum of the two rolls results in an even number. Let D be the event where each of the dice shows a different number and the sum of the two rolls results in an even number. Note that we have $6 \times 5 = 30$ outcomes in which the dice show different numbers; each of these has probability $\frac{1}{36}$ since we have a uniform probability distribution. Among these outcomes, the numbers add up to an even number if they are both odd—there are 6 of these:

$$(1, 3), (3, 1), (1, 5), (5, 1), (3, 5), (5, 3)$$

or if they are both even—there also 6 of these:

$$(2, 4), (4, 2), (2, 6), (6, 2), (4, 6), (6, 4)$$

So that's 12 outcomes of probability $\frac{1}{36}$ in D .

Let S be the event that where the two dice show the same number and the sum of the two rolls results in an even number. Each of these has probability $\frac{1}{36}$. In all these outcomes the numbers add up to an even number; hence we have another 6 outcomes of probability $\frac{1}{36}$ each in S .

Note that D and S partition E . That means:

$$\begin{aligned} \Pr[E] &= \Pr[D \cup S] \\ &= \Pr[D] + \Pr[S] \\ &= (6 + 6) \times \frac{1}{36} + 6 \times \frac{1}{36} \\ &= \frac{18}{36} \\ &= \frac{1}{2} \end{aligned}$$