CIS 160

Recitation Guide - Week 5

Topics Covered: Graph Induction, Probability, Conditional Probability

Problem 1:

In this problem we illustrate a common trap that we can fall in when proving statements about graphs by induction on the number of vertices or the number of edges. Here is a *false statement*: "If every vertex in a simple graph G has strictly positive (> 0) degree, then G is connected".

(a) Prove that the statement is indeed false by providing a counterexample.

(b) Since the statement is false, there must be something wrong in the following "proof". Pinpoint the *first* logical mistake (unjustified step).

Buggy Proof:

We prove the statement by induction on the number of vertices. Let $P(n)$ be the following proposition: "for any graph with n vertices, if every vertex has strictly positive degree, then the graph is connected".

Base Cases: Notice that $P(1)$ is vacuously true. We also show that $P(2)$ is true. Notice that there is only one graph with two vertices of strictly positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

Induction Hypothesis: Assume that for some $k \geq 2$, $P(k)$ is true.

Induction Step:

Consider a graph G_{old} with k vertices in which every vertex has strictly positive degree. By the Induction Hypothesis this graph is connected. Now we add one more vertex, call it u, to obtain a graph G_{new} with $k+1$ vertices.

All that remains is to check that in G_{new} there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G_{old}, which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G_{new}. ✓

(c) Now consider the changed Induction Step and identify a mistake in this proof.

Induction Step:

Consider a graph G with $k+1$ vertices in which every vertex has strictly positive degree. Remove an arbitrary vertex, call it u, and now we have a graph G' with k vertices. By the Induction Hypothesis this graph is connected. Now we add u back in to obtain a graph G with $k+1$ vertices.

All that remains is to check that in G there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G', which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G. ✓

Problem 2:

A standard 52-card deck consists of cards labelled 2 through 10, an Ace, Jack, Queen and King, each with four suits. A hand consists of five cards drawn from the deck. Richard is a wannabe magician who is trying to draw specific hands for his new magic show: The Appearing Pigeon. However, Richard can't quite consistently draw a specific hand, but he has learned how to draw any hand uniformly at random from the Gandhi school of magic.

(a) Calculate the probability that he draws a four of a kind successfully. A hand is considered "four of a kind" if it contains all four suits of a specific label.

(b) Calculate the probability that he draws a full house successfully. A hand is considered "full house" if it contains three cards of the same label and two cards of the some other label (i.e. 3 Aces and 2 8s).

Problem 3:

Compute the probability of the event "when we roll two identical 6-sided beige dice the numbers add up to an even number."