CIS 160

Recitation Guide - Week 6

Topics Covered: Graph Induction, Probability, Conditional Probability

Problem 1:

In this problem we illustrate a common trap that we can fall in when proving statements about graphs by induction on the number of vertices or the number of edges. Here is a *false statement*: "If every vertex in a simple graph G has strictly positive (> 0) degree, then G is connected".

(a) Prove that the statement is indeed false by providing a counterexample.

(b) Since the statement is false, there must be something wrong in the following "proof". Pinpoint the *first* logical mistake (unjustified step).

Buggy Proof:

We prove the statement by induction on the number of vertices. Let $P(n)$ be the following proposition: "for any graph with n vertices, if every vertex has strictly positive degree, then the graph is connected".

<u>Base Cases:</u> Notice that $P(1)$ is vacuously true. We also show that $P(2)$ is true. Notice that there is only one graph with two vertices of strictly positive degree, namely, the graph with an edge between the vertices, and this graph is connected.

<u>Induction Hypothesis:</u> Assume that for some $k \geq 2$, $P(k)$ is true.

<u>Induction Step:</u>

Consider a graph G_{old} with k vertices in which every vertex has strictly positive degree. By the Induction Hypothesis this graph is connected. Now we add one more vertex, call it u, to obtain a graph G_{new} with $k+1$ vertices.

All that remains is to check that in G_{new} there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G_{old}, which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G_{new}. ✓

(c) Now consider the changed Induction Step and identify a mistake in this proof.

<u>Induction Step:</u>

Consider a graph G with $k+1$ vertices in which every vertex has strictly positive degree. Remove an arbitrary vertex, call it u, and now we have a graph G' with k vertices. By the Induction Hypothesis this graph is connected. Now we add u back in to obtain a graph G with $k+1$ vertices.

All that remains is to check that in G there is a walk from u to every other vertex v. Since u has positive degree, there is an edge from u to some other vertex, say w. But w and v are in G', which is connected, and therefore there is a walk from w to v. This gives a walk $u - w - v$ in G. ✓

Solution:

(a) Consider the graph $G = (V, E)$ where $V = \{a, b, c, d\}$ and $E = \{\{a, b\}, \{c, d\}\}$. Every vertex has degree one, however the graph is not connected (there is no path from a to c, for example).

(b) The logical mistake in the proof is where we "add one more vertex" in the induction step. It is certainly possible to add one more vertex to a graph such that all vertices have strictly positive degree, but this constructs a *particular* type of graph G_{new} with $k+1$ vertices, where we actually had to show $P(k+1)$, which is that the claim holds for any graph with $k+1$ vertices. In particular, there are graphs with $k+1$ vertices where all its vertices have strictly positive degree that cannot be constructed from graphs with k vertices that fulfills the same condition. For instance, there does not exist any graph with 3 vertices where all its vertices have strictly positive degree such that by adding a new vertex we obtain graph G in part a). This highlights the importance of starting with an arbitrary graph with $k+1$ vertices, then deconstruct it to obtain a graph with k vertices to apply the IH to in graph induction proofs!

There are a couple of statements that may seem "bogus" but are actually not. They are as follows.

(a) "$P(1)$ is vacuously true": This is not "bogus", as a simple graph with 1 vertex must not have any edges, so it cannot have strictly positive degree.

(b) "Let k be an arbitrary integer such that $k \geq 2$": This is not "bogus", as we have an additional base case for $n = 2$, while $P(1)$ is proved separately.

(c) After removing a vertex, we have to make sure that in G', the properties specified in IH still exist. In this case, we have to make sure that after removing a vertex, every vertex still has a strictly positive degree to apply IH.

Consider the neighbors of u in G. If there was a neighbor x such that the degree of x in G was 1, since its only neighbor was removed, its degree in G' would be 0. Therefore, we cannot always apply IH to G'.

Problem 2:

A standard 52-card deck consists of cards labelled 2 through 10, an Ace, Jack, Queen and King, each with four suits. A hand consists of five cards drawn from the deck. Richard is a wannabe magician who is trying to draw specific hands for his new magic show: The Appearing Pigeon. However, Richard can't quite consistently draw a specific hand, but he has learned how to draw any hand uniformly at random from the Gandhi school of magic.

(a) Calculate the probability that he draws a four of a kind successfully. A hand is considered "four of a kind" if it contains all four suits of a specific label.

(b) Calculate the probability that he draws a full house successfully. A hand is considered "full house" if it contains three cards of the same label and two cards of the some other label (i.e. 3 Aces and 2 8s).

Solution:

(a) The sample space, Ω, is all the possible ways in which 5 cards can be chosen from the 52 card deck. $|\Omega| = \binom{52}{5}$. Since we are equally likely to pick any hand from the deck, note that the sample space is uniform.

Let A $\subseteq \Omega$ be the event (set of outcomes) where he draws a 4 of a kind. Since Ω is a uniform sample space, $\Pr[A] = \frac{|A|}{|\Omega|}$.

We can compute $|A|$ as follows:

Step 1: Pick a label. $\binom{13}{1}$ ways

Step 2: Pick the 4 cards having the same label. $\binom{4}{4}$ ways

Step 3: Pick the suit for the 5th card. $\binom{4}{1}$ ways

Step 4: Pick the label of the 5th card. $\binom{12}{1}$ ways (We already picked 4 cards of the same label, there are 12 labels left)

By the Multiplication Rule, $|A| = \binom{13}{1} \times \binom{4}{4} \times \binom{4}{1} \times \binom{12}{1}$. Thus, Pr[A] $= \frac{|A|}{|\Omega|} = \frac{\binom{13}{1}\times\binom{4}{4}\times\binom{4}{1}\times\binom{12}{1}}{\binom{52}{5}}$.

(b) Let $A \subseteq \Omega$ be the event (set of outcomes) where he draws a full house. Since Ω is a uniform sample space, $\Pr[A] = \frac{|A|}{|\Omega|}$.

We can compute $|A|$ as follows:

Step 1: Pick a label for the three of a kind. $\binom{13}{1}$ ways

Step 2: Pick the suits in that triple. $\binom{4}{3}$ ways

Step 3: Pick a label for the pair. $\binom{12}{1}$ ways

Step 4: Pick the suits for the pair. $\binom{4}{2}$ ways

By the Multiplication Rule, $|A| = \binom{13}{1} \times \binom{4}{3} \times \binom{12}{1} \times \binom{4}{2}$. Thus $\Pr[A] = \frac{|A|}{|\Omega|} = \frac{\binom{13}{1}\times\binom{4}{3}\times\binom{12}{1}\times\binom{4}{2}}{\binom{52}{5}}$.

Problem 3:

Compute the probability of the event "when we roll two identical 6-sided beige dice the numbers add up to an even number."

Solution:

We first describe the sample space for this problem by:

$$\Omega = \{\{x, y\} \mid x, y \in [1..6]\} = \{\{x, y\} \mid x, y, x \neq y\} \cup \{x\text{-}x \mid x \in [1..6]\}$$

Note that our probability distribution is not uniform.

Let E be the event where the sum of the two rolls results in an even number. Note that we have $\binom{6}{2} = 15$ outcomes in which the dice show different numbers; each of these has probability $2 \times \frac{1}{36} = \frac{1}{18}$. Among these outcomes, the numbers add up to an even number if they are both odd, and there are 3 of these, $\{1, 3\}, \{1, 5\}, \{3, 5\}$, or if they are both even – there also 3 of these: $\{2, 4\}, \{2, 6\}, \{4, 6\}$. So that's 6 outcomes of probability $\frac{1}{18}$ each in which the numbers are different.

We also have 6 more outcomes in which the die show the same number; each of these has probability $\frac{1}{36}$. In all these outcomes the numbers add up to an even number, hence we have another 6 outcomes of probability $\frac{1}{36}$ each.

We now calculate the desired probability using the definition of event:

$$\begin{aligned}
\Pr[E] &= \sum_{w \in E} \Pr(w) \\
&= \Pr[\{1, 3\}] + \Pr[\{1, 5\}] + \Pr[\{3, 5\}] + \Pr[\{2, 4\}] + \Pr[\{2, 6\}] + \Pr[\{4, 6\}] + \sum_{x \in [1..6]} \Pr[x\text{-}x] \\
&= 6 \times \frac{1}{18} + 6 \times \frac{1}{36} \\
&= \frac{1}{3} + \frac{1}{6} \\
&= \frac{1}{2}
\end{aligned}$$

Note that since "adding up to even" is an event in which the die color doesn't matter, we could have provided an equivalent solution that assumes the dice are green-purple rather than beige-beige. Doing so would change the sample space, which leads to the following alternate solution.

Alternate Solution:

We first define the sample space for this problem as:

$$\Omega = \{(x, y) \mid x, y \in [1..6]\} = \{(x, y) \mid x, y \in [1..6], x \neq y\} \cup \{(x, x) \mid x \in [1..6]\}$$

This sample space gives us a uniform probability distribution, where each outcome has probability $\frac{1}{36}$.

Let E be the event where the sum of the two rolls results in an even number.

Let D be the event where each of the dice shows a different number and the sum of the two rolls results in an even number. Note that we have $6 \times 5 = 30$ outcomes in which the dice show different numbers; each of these has probability $\frac{1}{36}$ since we have a uniform probability distribution. Among these outcomes, the numbers add up to an even number if they are both odd—there are 6 of these:

$$(1,3),(3,1),(1,5),(5,1),(3,5),(5,3)$$

or if they are both even—there also 6 of these:

$$(2,4),(4,2),(2,6),(6,2)(4,6),(6,4)$$

So that's 12 outcomes of probability $\frac{1}{36}$ in D.

Let S be the event that where the two dice show the same number and the sum of the two rolls results in an even number. Each of these has probability $\frac{1}{36}$. In all these outcomes the numbers add up to an even number; hence we have another 6 outcomes of probability $\frac{1}{36}$ each in S.

Note that D and S partition E. That means:

$$\begin{aligned}
\Pr[E] &= \Pr[D \cup S] \\
&= \Pr[D] + \Pr[S] \\
&= (6+6) \times \frac{1}{36} + 6 \times \frac{1}{36} \\
&= \frac{18}{36} \\
&= \frac{1}{2}
\end{aligned}$$

Note that because we have a uniform probability distribution, we also could have treated this solely as a counting problem, by applying the following formula:

$$\begin{aligned}
\Pr[E] &= \frac{|E|}{|\Omega|} \\
&= \frac{6+6+6}{36} \\
&= \frac{18}{36} \\
&= \frac{1}{2}
\end{aligned}$$