CIS 160 Recitation 6

Probability, Graph

October 7-8, 2021

Intro to Probability

- The sample space Ω is the set of all possible outcomes.
- The probability space is a sample space together with a probability distribution assigned to each outcome $\omega \in \Omega$ s.t.

$$0 \leq Pr[\omega] \leq 1$$

$$\sum_{\omega \in \Omega} Pr[\omega] = 1$$

- A subset of the sample space is called an event.
- For any event $A \in \Omega$, the probability of A is defined as:

$$Pr[A] = \sum_{\omega \in A} Pr[\omega]$$

Steps to Solve Probability Problems

1. Define a sample space Ω of the experiment.
2. Define the probability distribution.
3. Find the event of interest A (subset of outcomes $A \subseteq \Omega$ that are of interest).
4. Compute $Pr[A]$ by adding up probabilities of the outcomes in A.

The Inclusion-Exclusion Formula

- If A, B, C are any events,

$$\begin{aligned} Pr[A \cup B] &= Pr[A] + Pr[B] - Pr[A \cap B] \\ Pr[A \cup B \cup C] &= Pr[A] + Pr[B] + Pr[C] \\ &\quad - Pr[A \cap B] - Pr[A \cap C] - Pr[B \cap C] \\ &\quad + Pr[A \cap B \cap C] \end{aligned}$$

- Union-bound

$$Pr[\cup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} Pr[A_i]$$

If the events are pairwise disjoint, the inequality becomes equality.

Conditional Probability

- We are interested in event A given that we already know another event B has happened.
- $Pr[A|B]$ "The probability of A given B"

$$Pr[A|B] = \frac{Pr[A \cap B]}{Pr[B]}$$

- When $Pr[B] = 0$, $Pr[A|B]$ is undefined

Graph Lemmas

- The Handshaking Lemma: the sum of the degrees of all vertices in a graph is twice the number of edges
- In any graph, there are an even number of vertices of odd degree

Connected Components, Subgraphs

- G is **connected** if there is a path in G between its every pair of vertices.
- $H = (V, E)$ is a **connected component** (island) of G if:
  - H is a subgraph of G.
  - H is connected.
  - H is maximal (H is not contained in any other connected subgraph of G).
- $H = (V, E)$ is a **subgraph** of $G = (V, E)$ if $V \subseteq V$ and $E \subseteq E$.
- H is an **induced subgraph** of a graph G if $V_H \subseteq V_G$, and (u, v) is an edge in H iff (u, v) is an edge in G.

Acyclic Graphs, Trees, Forests

- A graph with no cycles is **acyclic**.
- A **tree** is a connected acyclic graph.
- A vertex of degree greater than 1 in a tree is called an **internal vertex**, otherwise it is called a **leaf**.
- A **forest** is an acyclic graph.
- Every tree with edges has at least 2 leaves.