CIS 160

Recitation Guide - Week 5

Topics Covered: Strong Induction, Pigeonhole Principle, Probability

Problem 1:

Consider any five points $P_1, ..., P_5$ in the interior of a square of length 2. Show that at least two of the points must be at a distance of at most $\sqrt{2}$ apart.

Solution:

Consider partitioning the square into four smaller squares, all of equal size (draw a line down from the middle of the top to the bottom edge and another from right to left from the middle of the left to the right edge). Now, each of these squares has side length 1, so the diagonal of each square will be of length $\sqrt{2}$, which is the maximum distance between any two points in the square. Let the pigeons be the points, and the squares be the holes. By the Pigeonhole Principle, we have that there exists at least $\lceil \frac{5}{4} \rceil = 2$ points in one square. Since the maximum distance between any two points within the same square is $\sqrt{2}$, these two points will be at a distance of at most $\sqrt{2}$ apart.

Problem 2:
A car needs 1 unit of length to park while a truck needs 2 units of length. Assume that cars are indistinguishable and so are trucks. How many distinct car/truck parking patterns are possible along an n unit long sidewalk? Prove your result.

Solution:

We write the parking patterns as a string of C's and T's. Here are two distinct ways in which 3 cars and 2 trucks can be parked along a sidewalk that is 7 units long: CTCCT and TCTCC.

length	**patterns**	#
1	C	1
2	CC T	2
3	CT CCC TC	3
4	CCT TT CTC CCCC TCC	5
5	CTT CCCT TCT CCTC TTC CTCC CCCCC TCCC	8

We prove by induction that the number of distinct parking patterns along a sidewalk of length $n \geq 1$ is F_{n+1}, where F_{n+1} refers to the $n+1'$st number in the Fibonacci sequence.

$$F(n) = \begin{cases} 0 & n = 0 \\ 1 & n = 1 \\ F(n-1) + F(n-2) & n > 1 \end{cases}$$

Base Case 1: $n = 1$. Only 1 pattern, C. $F_2 = 1$. ✓

Base Case 2: $n = 2$. 2 patterns, CC and T. $F_3 = 2$. ✓

Induction Hypothesis: Assume that the number of patterns for length j is F_{j+1} where $1 \leq j \leq k$, $j \in \mathbb{Z}^+$, for some $k \in \mathbb{Z}^+$.

Induction Step: Let k arbitrary, $k \in \mathbb{Z}, k \geq 2$.

Now consider a pattern p for length $k+1$. Depending on whether this pattern ends with a car or a truck, we have two cases.

Case 1. The last vehicle in p is a car. Let r be the string of vehicles before the last vehicle in p. Then, $p = r$C. Therefore, r has length $k + 1 - 1 = k$. By IH, there are F_{k+1} distinct r's. Therefore, in this case, we have F_{k+1} distinct patterns.

Case 2. The last vehicle in p is a truck. Let s be the string of vehicles before the last vehicle in p. Then, $p = s$T. Therefore, s has length $k + 1 - 2 = k - 1$. By IH, there are F_k distinct s's therefore F_k distinct p's in this case.

Since these two cases are disjoint, by the addition rule, there are $F_{k+1} + F_k = F_{k+2}$ distinct patterns. Thus, we have concluded our induction step and our proof.

Problem 3:
Compute the probability of the event "when we roll n (distinguishable) fair dice, any k of the dice show the same number while the other $n-k$ show numbers different from the one shown by the k dice." Assume $n \geq 3$ and $\frac{n}{2} < k < n$.

Solution:

As discussed in class, we have a uniform probability distribution whose outcomes are sequences of length n of numbers from $[1..6]$. In other words, the sample space Ω is given by the cartesian product of $[1..6] \times \cdots \times [1..6]$ (n times), i.e., $\Omega = [1..6]^n$. By the Multiplication Rule, there are $6 \times \cdots \times 6 = 6^n$ such sequences so each outcome has probability $\dfrac{1}{6^n}$.

Let E be the event where exactly k of the dice show the same number. We see that we are trying to find $\Pr[E]$. To compute the desired probability it suffices to count the cardinality of E, i.e., the number of sequences (of interest) in which k positions have the same number from $t \in [1..6]$ while the other $n-k$ position show numbers different from t. Such a sequence can be constructed as follows:

Step 1: Choose $t \in [1..6]$. This can be done in 6 ways.
Step 2: Choose k of the n positions in the sequence. This can be done in $\binom{n}{k}$ ways.
Step 3: Place t in each of these positions. This can be done in 1 way.
Step 4: For each of the remaining $n-k$ positions choose a number from $[1..6] \setminus \{t\}$. We see that there are 5 such numbers, and $n-k$ positions that we have left to fill. Thus, this can be done in 5^{n-k} ways.

By the Multiplication Rule, the number of sequences of interest is $6\binom{n}{k}5^{n-k}$. Hence, the probability we are asked for is given by:

$$\Pr[E] = \frac{|E|}{|\Omega|} = \left(6\binom{n}{k}5^{n-k}\right)\frac{1}{6^n} = \binom{n}{k}\frac{5^{n-k}}{6^{n-1}}$$

Aside:

A student from recitation brought up the following point. Why doesn't our method overcount? For example, let us consider the outcome $T = (1, 1, 2, 2, 3, 3) \in \Omega$ for $n = 6$. We could derive T by first picking $t = 1$ and assigning it to the first two dice. Then in step 4 we could generate $(2, 2, 3, 3)$. Similarly, our method allows us to pick $t = 3$, assign it to the last two dice, and place $(1, 1, 2, 2)$ in the first four dice.

To see why this cannot happen, look at the assumption that $\frac{n}{2} < k < n$. We will now show that each event will only be counted once in our counting procedure. Let us consider any tuple T of length n which is in Ω. By our assumption more than half of these positions are filled by a number between 1 and 6. WLOG assume that this number is 1. We will show that it is impossible to construct this tuple again. First, the remaining $n-k$ positions of this tuple will be uniquely generated by step 4 so we don't have to worry about step 4 causing a repetition of this tuple.

Now let us consider choosing a different $t \neq 1$. We will show it is impossible to construct our tuple T again when $t \neq 1$. To see this, notice that because $k > \frac{n}{2}$ any tuple constructed with t chosen in step 1 will have more than half of its positions filled with t. But our tuple T had 1 in more than half of its positions! So no matter what we do in steps 2-4 we could have never constructed T again.