CIS 160 Recitation #5

Strong Induction and Probability

Strong Induction

Let $P(n)$ be a predicate whose truth depends on n.

We want to prove that $P(n)$ is true for all integers n greater than or equal to some integer n_0 .

Base Case: Show $P(n_0)$ is true.

Induction Hypothesis: Assume $P(n_0), P(n_0 + 1), ..., P(k)$ are true for some integer $k \geq n_0$

Induction Step: Using these assumptions, prove that $P(k + 1)$ is true.

Probability Definitions

- Sample space (Ω) - set of all outcomes of an experiment/random process
- Probability space - a sample space with a probability distribution- a mapping of outcomes $\omega \in \Omega$ to probabilities such that:

$$0 \leq \Pr[\omega] \leq 1$$

$$\sum_{\omega \in \Omega} \Pr[\omega] = 1$$

- An event is a subset of the sample space
  - The probability of an event A is defined as:

$$\Pr[A] = \sum_{\omega \in A} \Pr[\omega]$$

Probability (continued)

- If the probability distribution is **uniform** (if every outcome in the sample space is equally likely), then:

$$\forall \omega \in \Omega, \Pr[\omega] = \frac{1}{|\Omega|} \qquad A \subseteq \Omega, \Pr[A] = \frac{|A|}{|\Omega|}$$

- Inclusion-Exclusion Formula:

$$A, B \subseteq \Omega, \Pr[A \cup B] = \Pr[A] + \Pr[B] - \Pr[A \cap B]$$

For events $A_1, A_2, \ldots, A_n$ in some probability space, let $S_1 = \{(i_1) | 1 \leq i_1 \leq n\}$, $S_2 = \{(i_1, i_2) | 1 \leq i_1 < i_2 \leq n\}$, and more generally let $S_p = \{(i_1, i_2, \ldots, i_p) | 1 \leq i_1 < i_2 < \ldots \leq i_p \leq n\}$. Then we have

$$\Pr[\cup_{i=1}^{n} A_i] = \sum_{i \in S_1} \Pr[A_i] - \sum_{(i_1,i_2) \in S_2} \Pr[A_{i_1} \cap A_{i_2}] + \sum_{(i_1,i_2,i_3) \in S_3} \Pr[A_{i_1} \cap A_{i_2} \cap A_{i_3}] - \cdots + (-1)^{n-1} \Pr[\cap_{x=1}^{n} A_x]$$

$$\Pr[\cup_{i=1}^{n} A_i] \leq \sum_{i=1}^{n} \Pr[A_i]$$