Problem 1: Suppose we have the following sequence:

\[ a_1 = 1 \quad a_2 = 3 \quad a_i = a_{i-2} + 2a_{i-1}, \quad i \in \mathbb{Z}, \ i \geq 3 \]

Use strong induction to prove that for all integers \( n \geq 1 \), \( a_n \) is odd.

**Base Case:** \( n = 1 \), \( a_1 = 1 \), \( \sqrt{\text{because } 1 \text{ is odd}} \)

\( a_2 = 3 \), \( \sqrt{\text{because } 3 \text{ is odd}} \)

**IH:** Assume \( a_k \) is true for some \( k \geq 1 \)

**IS:** We want to show that \( a_{k+1} \) is odd.

\[ a_{k+1} = a_{k-1} + 2a_k \]

By IH, we know that \( a_k \) is odd, so let \( a_k = 2p+1 \), for some \( p \in \mathbb{N}^+ \)

By IH, we know that \( a_{k-1} \) is odd, then \( a_{k-1} = 2q+1 \), for some \( q \in \mathbb{N} \)

\[ a_{k+1} = 2q+1 + 2(2p+1) \]

\[ = 2q+1 + 4p+2 \]

\[ = 2q + 4p + 2 + 1 \]

\[ = 2(q + 2p + 1) + 1 \]

\[ \Rightarrow 2m + 1 \quad \text{where } m \in \mathbb{Z} = q + 2p + 1 \]

\[ \checkmark \]
Problem 2: Anusha and Brandon are playing a game in which there are two non-empty bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Anusha and Brandon’s current configuration, both bags initially start with the same number of marbles. Prove that one of them can guarantee a win.

Winning strategy: Pick same amt as prev move from the other bag.

Base case: n=1. First player picks a marble from bag 1. 2nd player wins.

IH: Assume P2 has a winning strategy for n=j, where 1 ≤ j ≤ k, for some k ≥ 1.

IS: WTS that P2 has winning strategy for n=k+1.

\[ \begin{array}{c|c}
0 & 0 \\
\hline
1 & 2 \\
\end{array} \rightarrow k+1 marbles in each bag.

P1 can take anywhere from 1 to k+1 marbles.

Case 1: P1 takes k+1 marbles from one bag.

In this case P2 winning strategy is to take k+1 marbles from the other bag.

Case 2: P1 takes k marbles from one bag

P2 also takes from other bag

2 bags w/ one marble each

Case 3: P1 takes i marbles, 1 ≤ i ≤ k

P2 also take from other bag

we are left w/ 2 bags w/ k+1-i marbles where \( \frac{1}{2} ≤ k+1-i ≤ k \)

By IH, P2 has a winning strategy for k+1 marbles in each bag.

\[ \sqrt{2} \]
Problem 3:

Let \( S \) be a set of 16 distinct positive integers such that \( \forall x \in S, \ x < 60 \). Show that there exist distinct integers \( a, b, c, d \in S \) such that \( a + b = c + d \).

We are finding 2 pairs in \( S \) w/ same sum.

We use PHP:

- **Pigeons**: pairs of numbers from set \( S \)
- **Holes**: Sums themselves
- \( \# \) pigeons: \( \binom{16}{2} = 120 \)
- \( \# \) holes: all possible sums
  - Smallest possible sum = 3
  - \( \lambda \in S \)
  - \( 1 \leq \lambda \leq 59 \)
  - Largest possible sum = 117
  - \( \sum \in [3..117] \) \( \Rightarrow \) 115 possible sums

By PHP, there exists a hole w/ \( \left\lfloor \frac{120}{115} \right\rfloor = 2 \) pigeons, i.e., there exists 2 pairs w/ same sum.

Assume for contradiction that \( a + b = a + c \Rightarrow b = c \Rightarrow (a, b) (a, c) \) are the same pair. \( \emptyset \)

Therefore, no overlap \( \Rightarrow \) 2 pairs w/ distinct \( \# \)'s.