CIS 160

Recitation Guide - Week 5

Topics Covered: Strong Induction, Pigeonhole Principle

Problem 1: Suppose we have the following sequence:

$$a_1 = 1 \qquad a_2 = 3 \qquad a_i = a_{i-2} + 2a_{i-1}, \quad i \in \mathbb{Z}, i \geq 3$$

Use strong induction to prove that for all integers $n \geq 1$, a_n is odd.

Solution:

Define $P(k)$ to be the claim that a_k is odd.

<u>Induction Hypothesis:</u> Assume $P(j)$ is true, for $1 \leq j \leq k$, for some $k \in \mathbb{Z}$, $k \geq 1$

<u>Base Case 1:</u> When $n = 1$, $a_1 = 1$, which is odd. ✓

<u>Base Case 2:</u> When $n = 2$, $a_2 = 3$, which is also odd. ✓

<u>Induction Step:</u> Let k be arbitrary, $k \in \mathbb{Z}, k \geq 2$.

We want to show that $P(k+1)$ is true. We know $a_{k+1} = a_{k-1} + 2a_k$. By the Induction Hypothesis, a_{k-1} and a_k are both odd. Let $a_{k-1} = 2x + 1$ and $a_k = 2y + 1$, for some integers x and y. Hence,

$$\begin{aligned} a_{k+1} &= (2x + 1) + 2(2y + 1) \\ &= 2x + 1 + 4y + 2 \\ &= 2(x + 2y + 1) + 1 \end{aligned}$$

Which is odd because $x + 2y + 1 \in \mathbb{Z}$

Problem 2: Anusha and Brandon are playing a game in which there are two non-empty bags with an equal number of marbles in them. In this game, the two players take turns removing marbles from one of the bags. In each turn, the player can remove any positive number of marbles as long as they are all from the same bag. The winner of the game is the player that removes the last marble. In Anusha and Brandon's current configuration, both bags initially start with the same number of marbles. Prove that one of them can guarantee a win.

Solution:

Consider the following strategy: the player who goes second always removes the same number of marbles as the player who went first, but from the other bag. If Brandon goes first, Anusha can always win by using this strategy.

Define $P(n)$ to be the claim that this strategy always works for bags that start with n marbles each. We prove our strategy works by strong induction.

Induction Hypothesis: Assume $P(j)$ is true, for $1 \leq j \leq k$, for some $k \in \mathbb{Z}^+$.

Base Case: $k = 1$. Brandon goes first. His only move is to remove one marble from a bag. Anusha then removes the last marble from the other bag. Thus the strategy works.

Induction Step: We want to show that the claim still holds if each bag has $k + 1$ marbles. So, we start with two bags containing $k + 1$ marbles each. In Brandon's first move, he can remove m number of marbles for $m \in \mathbb{Z}$, $1 \leq m \leq k + 1$.

Case 1: $m = k + 1$ (i.e., Brandon removes all the marbles from a bag).

In this case, Anusha can just take the $k + 1$ marbles in the other bag. Because she took the last marble, she wins.

Case 2: $1 \leq m \leq k$:

Thus, after the first move, the bags contain $k+1-m$ and $k+1$ marbles. According to the strategy, Anusha removes m marbles from the other bag so that both bags now contain $k + 1 - m$ marbles. We can view the current state of the game as a new game in which both piles contain $k + 1 - m$ marbles. Since $1 \leq k+1-m \leq k$, we can apply the Induction Hypothesis to state that this strategy will always work.

Problem 3:

Let S be a set of 16 distinct positive integers such that $\forall x \in S$, $x < 60$. Show that there exist distinct integers $a, b, c, d \in S$ such that $a + b = c + d$.

Solution:

We will prove this using the Pigeonhole Principle.

Every pair of integers will have an associated sum, and there are $\binom{16}{2} = 120$ unordered pairs of distinct elements in S. Since all elements of S are between 1 and 59 inclusive, the sum of any pair of distinct elements will be between 3 and 117 inclusive, which gives 115 possibilities.

Let the unordered pairs represent the pigeons and the possible sums represent the holes. Since there are 120 pigeons and 115 holes, by PHP there exist $\lceil 120/115 \rceil = 2$ distinct pairings that map to the same sum.

However, we are not quite done yet. What if the unordered pairs overlap? If the 2 inputs that map to the same sum are $\{a, b\}$ and $\{a, c\}$ (with distinct a, b, c), then this would be invalid. This would imply, however, that $a + b = a + c \implies b = c$, which contradicts the fact that a, b, c are distinct. Thus, the two pairings that have the same sum have no overlaps.