CIS 160

# Recitation Guide - Week 4

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**Topics Covered:** Induction, Pigeonhole Principle

**Problem 1:**

**All the sheep in Arnold's flock have the same color!** Arnold claims that he can use induction to prove all sheep in the world have the same color. Find the fault in his reasoning.

Base Case: Size = 1. One sheep, one color. ✓

Induction Hypothesis: Assume that in a flock of size $k$, where $k \in \mathbb{Z}^+$, all sheep have the same color.

Induction Step: We want to prove the claim is true for a flock of size $k+1$. Take one sheep, let's call it Dolly, out. What remains is a flock of size $k$, so by IH, they all share the same color. Now put Dolly back in and take out another sheep, let's call it Polly, out. By IH, what remains is a flock of size $k$, so by IH, they all share the same color. Dolly and Polly must share the same color as they both are the same color as the other $k-1$ sheep. Thus we arrive at the conclusion that all $k+1$ sheep share the same color.

**Solution:**

The problem lies in the step where we try to show that having one sheep of the same color implies that two sheep must have the same color (That is, when $k = 1$ and $k + 1 = 2$). It is possible that the sheep don't share the same color. When we take Dolly out, we can use the IH to say that Polly has the same color as herself. Similarly, when we take Polly out, by IH Dolly also has the same color as herself. Notice here that the two subsets of sheep that we applied the IH on are disjoint. Since there is no common element between the two subsets, we cannot conclude that the color of one subset is the same as the color of the other subset.

Thus, the induction used in the question is not valid.

**Problem 2:**

Prove using induction that for any positive integer $n$ and for any integers $d_0, d_1, \ldots, d_{n-1} \in [0..9]$ we have:

$$\sum_{j=0}^{n-1} d_j \cdot 10^j < 10^n$$

**Solution:**

Base Case: $n = 1$.

$$\sum_{j=0}^{0} d_j \cdot 10^j = d_0 \cdot 10^0 = d_0 < 10 = 10^1$$

$d_j s$ can only take values from 0 through 9, thus $d_0 < 10$. This concludes the Base Case.

Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 1$ such that:

$$\sum_{j=0}^{k-1} d_j \cdot 10^j < 10^k$$

Induction Step: We now want to show that our claim holds when $n = k + 1$. In other words, we seek to show that:

$$\sum_{j=0}^{k} d_j \cdot 10^j < 10^{k+1}$$

We see that we can show this as follows:

$$\begin{aligned}
\sum_{j=0}^{k} d_j \cdot 10^j &= d_k \cdot 10^k + \sum_{j=0}^{k-1} d_j \cdot 10^j && \text{(splitting the sum)} \\
&< d_k \cdot 10^k + 10^k && \text{(by IH)} \\
&= (d_k + 1) \cdot 10^k \\
&\leq (9 + 1) \cdot 10^k \\
&= 10^{k+1}
\end{aligned}$$

Thus, we have shown our claim is true when $n = k + 1$, concluding our Induction Step and completing our proof.

**Problem 3:**

Let $S$ be a set of 16 distinct positive integers such that $\forall x \in S,\ x < 60$. Show that there exist distinct integers $a, b, c, d \in S$ such that $a + b = c + d$.

**Solution:**

We will prove this using the Pigeonhole Principle.

Every pair of integers will have an associated sum, and there are $\binom{16}{2} = 120$ unordered pairs of distinct elements in $S$. Since all elements of $S$ are between 1 and 59 inclusive, the sum of any pair of distinct elements will be between 3 and 117 inclusive, which gives 115 possibilities.

Let the unordered pairs represent the pigeons and the possible sums represent the holes. Since there are 120 pigeons and 115 holes, by PHP there exist $\lceil 120/115 \rceil = 2$ distinct pairings that map to the same sum.

However, we are not quite done yet. What if the unordered pairs overlap? If the 2 inputs that map to the same sum are $\{a, b\}$ and $\{a, c\}$ (with distinct $a, b, c$), then this would be invalid. This would imply, however, that $a + b = a + c \implies b = c$, which contradicts the fact that $a, b, c$ are distinct. Thus, the two pairings that have the same sum have no overlaps.