CIS 160

Recitation Guide - Week 4

Topics Covered: Induction, Sticks and Crosses, Combinatorial Proofs

Problem 1:

All the sheep in Tien's flock have the same color! Tien claims that he can use induction to prove all sheep in the world have the same color. Find the fault in his reasoning.

Base Case: size = 1. One sheep, one color. ✓

Induction Hypothesis: Assume that in a flock of size k, where $k \in \mathbb{Z}^+$, all sheep have the same color.

Induction Step: We want to prove the claim is true for a flock of size $k+1$. Take one sheep, let's call her Winnie, out. What remains is a flock of size k, so by IH, they all share the same color. Now put Winnie back in and take out another sheep, let's call him Nicky, out. By IH, what remains is a flock of size k, so by IH, they all share the same color. Winnie and Nicky must share the same color as they both are the same color as the other $k-1$ sheep. Thus we arrive at the conclusion that all $k+1$ sheep share the same color.

Solution:

The problem lies in the step where we try to show that having one sheep of the same color implies that two sheep must have the same color (That is, when $k = 1$ and $k + 1 = 2$). It is possible that the sheep don't share the same color. When we take Winnie out, we can use the IH to say that Nicky has the same color as himself. Similarly, when we take Nicky out, by IH, Winnie also has the same color as herself. Notice here that the two subsets of sheep that we applied the IH on are disjoint. Since there is no common element between the two subsets, we cannot conclude that the color of one subset is the same as the color of the other subset.

Thus, the induction used in the question is not valid.

Problem 2:

A janitor needs to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he discovers that he has run out of supplies. Most importantly, he does not remember when his supplies ran out (meaning he could have used all his supplies in the first room). He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

How many ways could he have distributed the toilet paper rolls and soap bars to the different rooms?

Solution:

We can break this problem down into separate sticks and crosses problems and combine them at the end.

There are 6 rooms in which we distribute 10 toilet paper rolls. Arrange 10 crosses in a row. These crosses represent the toilet paper rolls. Since the toilet paper rolls are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ sticks to represent the rooms. Place the 5 sticks between some of the crosses (toilet paper rolls). The sticks would then separate the crosses into 6 parts, each of which represent one room. The number of arrangements of toilet paper rolls is then

$$\binom{10+6-1}{10} = \binom{15}{10}$$

We can do the same for the soap bars. 10 indistinguishable soap bars distributed along 6 rooms gives us

$$\binom{10+6-1}{10} = \binom{15}{10}$$

But this isn't the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the toilet paper rolls is distributed is independent of the order in which the soap bars are distributed.

Step 1: Choose a way to distribute the toilet paper rolls. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose a way to distribute the soap bars. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \boxed{\binom{15}{10}^2}$$

Problem 3:

Give a combinatorial proof for the following, where $m \leq n$:

$$\sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k} = 2^m \binom{n}{m}$$

Solution:

Consider the following counting problem:

> Given a set of n people, how many ways are there to select m members to gain membership to Penn's Premier Pigeon Catching Club, and then select any number of those members to be part of the executive board?

<u>RHS:</u> We use two steps to select people for the club and designate executive board members as follows.

> Step 1: Choose the m members who are selected out of the n people.
> Step 2: Designate any number of the m members as executive board members.

In Step 1, we are simply choosing m out of n items. Thus there are $\binom{n}{m}$ ways to do Step 1. In Step 2, we are taking a subset of m items, so there are 2^m ways to do Step 2. Applying the multiplication rule, there are $2^m\binom{n}{m}$ ways to accept members and designate executive board members, which is the RHS.

<u>LHS:</u> Let S be a set that includes all of the ways that we can choose members and designate executive board members. We can partition S into sets $S_0, S_1, S_2, \ldots, S_m$ where set S_k $(0 \leq k \leq m)$ represents all of the ways that we can accept m people and designate exactly k executive board members. For each k, $|S_k|$ can be calculated as follows.

> Step 1: Choose the k executive board members that we want from the n total people.
> Step 2: Select regular club members from the remaining $n - k$ people so that we end up with a total of m people.

In Step 1, we are simply choosing k people out of n, so there are $\binom{n}{k}$ ways to do Step 1. In Step 2, we must select regular club members so that we have a total of m regular club members. Since we have already accepted k executive board members, we can only accept $m - k$ more people to make up the club. In addition, we cannot choose any of those k executive board members to accept (since they have already been selected), so there are $n - k$ people to choose from. Thus, there are $\binom{n-k}{m-k}$ ways to do Step 2.

Applying the multiplication rule, there are $\binom{n}{k}\binom{n-k}{m-k}$ ways to select m members and designate exactly k executive board members.

Thus, the total number of ways to select m members and designate any number of executive board members is

$$|S| = \sum_{k=0}^{m} |S_k| = \sum_{k=0}^{m} \binom{n}{k}\binom{n-k}{m-k}$$

which is the LHS.

Problem 4:

Prove using induction that for any positive integer n and for any integers $d_0, d_1, \ldots, d_{n-1} \in [0..9]$ we have:

$$\sum_{j=0}^{n-1} d_j \cdot 10^j < 10^n$$

Solution:

Base Case: $n = 1$.

$$\sum_{j=0}^{0} d_j \cdot 10^j = d_0 \cdot 10^0 = d_0 < 10 = 10^1$$

$d_j s$ can only take values from 0 through 9, thus $d_0 < 10$. This concludes the Base Case.

Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 1$ such that:

$$\sum_{j=0}^{k-1} d_j \cdot 10^j < 10^k$$

Induction Step: We now want to show that our claim holds when $n = k + 1$. In other words, we seek to show that:

$$\sum_{j=0}^{k} d_j \cdot 10^j < 10^{k+1}$$

We see that we can show this as follows:

$$\begin{aligned}
\sum_{j=0}^{k} d_j \cdot 10^j &= d_k \cdot 10^k + \sum_{j=0}^{k-1} d_j \cdot 10^j && \text{(splitting the sum)} \\
&< d_k \cdot 10^k + 10^k && \text{(by IH)} \\
&= (d_k + 1) \cdot 10^k \\
&\leq (9 + 1) \cdot 10^k \\
&= 10^{k+1}
\end{aligned}$$

Thus, we have shown our claim is true when $n = k + 1$, concluding our Induction Step and completing our proof.