CIS 160

Recitation Guide - Week 3

Topics Covered: Sticks and Crosses, Combinatorial Proof, Multiset

Problem 1:

A janitor needs to distribute soap bars and toilet paper to customers of the hotel. He starts his shift with 10 bars of soap and 10 rolls of toilet paper. After the 6th room, he discovers that he has run out of supplies. Most importantly, he does not remember when his supplies ran out (meaning he could have used all his supplies in the first room). He cannot tell the difference between any two toilet paper rolls and between any two soap bars. However, he can easily tell the difference between toilet paper and soap bars.

(a) How many ways could he have distributed the toilet paper rolls and soap bars to the different rooms?

(b) How many ways could he have distributed the toilet paper rolls and soap bars given that there is at least one toilet paper roll and one soap bar in each room?

Solution:

(a) We can break this problem down into separate sticks and crosses problems and combine them at the end.

There are 6 rooms in which we distribute 10 toilet paper rolls. Arrange 10 crosses in a row. These crosses represent the toilet paper rolls. Since the toilet paper rolls are indistinguishable, their ordering is irrelevant. We now also have $6 - 1 = 5$ sticks to represent the rooms. Place the 5 sticks between some of the crosses (toilet paper rolls). The sticks would then separate the crosses into 6 parts, each of which represent one room. The number of arrangements of toilet paper rolls is then

$$\binom{10+6-1}{10} = \binom{15}{10}$$

We can do the same for the soap bars. 10 indistinguishable soap bars distributed along 6 rooms gives us

$$\binom{10+6-1}{10} = \binom{15}{10}$$

But this isn't the end! We still have to combine them. We can do this by using the multiplication rule, since the order in which the toilet paper rolls is distributed is independent of the order in which the soap bars are distributed.

Step 1: Choose a way to distribute the toilet paper rolls. From above, there are $\binom{15}{10}$ ways.

Step 2: Choose a way to distribute the soap bars. Again, there are $\binom{15}{10}$ ways.

which gives us

$$\binom{15}{10} \times \binom{15}{10} = \boxed{\binom{15}{10}^2}$$

(b) If there must be at least one toilet paper roll and one soap bar in each room, we can choose to put one toilet paper roll and one soap bar in each room to begin with. Since the soap bars are indistinguishable, there is only 1 way to perform this first step. Because toilet paper rolls and soap bars are indistinguishable, there is only 1 way to do this. Now, there are 4 rolls of toilet paper and 4 soap bars left to be placed in 6 rooms. Similar to above, using sticks and crosses along with Multiplication Rule, we have:

$$1 \times \binom{9}{5} \times \binom{9}{5} = \boxed{\binom{9}{5}^2}$$

Problem 2:

Give a combinatorial proof for the following, where $m \leq n$:

$$\sum_{k=0}^{m} \binom{n}{k} \binom{n-k}{m-k} = 2^m \binom{n}{m}$$

Solution:

Consider the following counting problem:

> Given a set of n people, how many ways are there to select m members to gain membership to Penn's Premier Pigeon Catching Club, and then select any number of those members to be part of the executive board?

<u>RHS:</u> We use two steps to select people for the club and designate executive board members as follows.

> Step 1: Choose the m members who are selected out of the n people.
> Step 2: Designate any number of the m members as executive board members.

In Step 1, we are simply choosing m out of n items. Thus there are $\binom{n}{m}$ ways to do Step 1. In Step 2, we are taking a subset of m items, so there are 2^m ways to do Step 2. Applying the multiplication rule, there are $2^m \binom{n}{m}$ ways to accept members and designate executive board members, which is the RHS.

<u>LHS:</u> Let S be a set that includes all of the ways that we can choose members and designate executive board members. We can partition S into sets $S_0, S_1, S_2, \ldots, S_m$ where set S_k $(0 \leq k \leq m)$ represents all of the ways that we can accept m people and designate exactly k executive board members. For each k, $|S_k|$ can be calculated as follows.

> Step 1: Choose the k executive board members that we want from the n total people.
> Step 2: Select regular club members from the remaining $n - k$ people so that we end up with a total of m people.

In Step 1, we are simply choosing k people out of n, so there are $\binom{n}{k}$ ways to do Step 1. In Step 2, we must select regular club members so that we have a total of m regular club members. Since we have already accepted k executive board members, we can only accept $m - k$ more people to make up the club. In addition, we cannot choose any of those k executive board members to accept (since they have already been selected), so there are $n - k$ people to choose from. Thus, there are $\binom{n-k}{m-k}$ ways to do Step 2.

Applying the multiplication rule, there are $\binom{n}{k}\binom{n-k}{m-k}$ ways to select m members and designate exactly k executive board members.

Thus, the total number of ways to select m members and designate any number of executive board members is

$$|S| = \sum_{k=0}^{m} |S_k| = \sum_{k=0}^{m} \binom{n}{k} \binom{n-k}{m-k}$$

which is the LHS.

Problem 3 (if time allows):

How many 5 letter sequences can be made from the letters in the word "PIAZZA"?

Solution:

The letters from the word "PIAZZA" can be represented as the multiset $\{2 \cdot A, 1 \cdot I, 1 \cdot P, 2 \cdot Z\}$

There are then 4 cases for the 5 letters to be chosen for the word, because exactly one letter must be left out from the multiset:

> Case 1: P is left out. In this case, our multiset of letters to choose from is $\{2 \cdot A, 1 \cdot I, 2 \cdot Z\}$ and there are $\frac{5!}{2!1!2!}$ sequences of letters.
>
> Case 2: I is left out. Similar to Case 1, there are $\frac{5!}{2!1!2!}$ sequences of letters.
>
> Case 3: A is left out. In this case, our multiset of letters to choose from is
>
> $\{1 \cdot A, 1 \cdot I, 1 \cdot P, 2 \cdot Z\}$ and there are $\frac{5!}{1!1!1!2!}$ sequences of letters.
>
> Case 4: Z is left out. Similar to Case 3, there are $\frac{5!}{1!1!1!2!}$ sequences of letters.

Thus, in total, there are $\frac{5!}{2!1!2!} + \frac{5!}{2!1!2!} + \frac{5!}{1!1!1!2!} + \frac{5!}{1!1!1!2!} = 180$ sequences of letters from the letters in "PIAZZA."

Alternate Solution:

We can also approach the problem in the following way. Let S be the set of 5 letter sequences from the letters in "PIAZZA." We can then count the number of 6-letter sequences in the following way:

> Step 1: Choose a 5 letter sequence from the letters in the word "PIAZZA." This can be done in $|S|$ ways.
>
> Step 2: Choose a letter to be the final letter in our sequence. Since we have already used 5 letters in step 1, this can only be done in 1 way.

Therefore, we can count the number of 6 letter sequences in $|S| \times 1 = |S|$ ways.

We can also compute the number of sequences of 6 letters as the number of permutations of the multiset $\{2 \cdot A, 1 \cdot I, 1 \cdot P, 2 \cdot Z\}$. From lecture, we know this can be done in $\frac{6!}{2!1!1!2!} = 180$ ways.

Since we have simply counted the same problem in two different ways, we get $|S| = 180$.