CIS 160

Recitation Guide - Week 3

Topics Covered: Set Proofs, PIE, Irrationality Proofs

Problem 1: Prove that

$$(A \cup B) \setminus C = (A \setminus C) \cup (B \setminus C)$$

Solution:

First, we show that $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

Let $x \in (A \cup B) \setminus C$ be arbitrary but particular. We want to show

$$x \in (A \cup B) \setminus C \implies x \in (A \setminus C) \cup (B \setminus C)$$

Since $x \in A \vee x \in B$, at least one of the following must be true:

Case 1: $x \in A \wedge x \notin C$. Thus, $x \in (A \setminus C)$ and so $x \in (A \setminus C) \cup (B \setminus C)$.

Case 2: $x \in B \wedge x \notin C$. Thus, $x \in (B \setminus C)$ and so $x \in (A \setminus C) \cup (B \setminus C)$.

In any case, $x \in (A \setminus C) \cup (B \setminus C)$, so $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$.

Now we show that $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$.

Let $x \in (A \setminus C) \cup (B \setminus C)$ be arbitrary but particular. We want to show

$$x \in (A \setminus C) \cup (B \setminus C) \implies x \in (A \cup B) \setminus C$$

At least one of the following is true:

Case 1: $x \in A \wedge x \notin C$. Since $x \in A$, $x \in (A \cup B)$. Thus, $x \in (A \cup B) \setminus C$.

Case 2: $x \in B \wedge x \notin C$. Since $x \in B$, $x \in (A \cup B)$. Thus, $x \in (A \cup B) \setminus C$.

In any case, $x \in (A \cup B) \setminus C$, so $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$.

Since $(A \cup B) \setminus C \subseteq (A \setminus C) \cup (B \setminus C)$ and $(A \setminus C) \cup (B \setminus C) \subseteq (A \cup B) \setminus C$, the two sets are equal.

Problem 2:

Given numbers 1 to 9, how many permutations of the numbers do not have at least 7 consecutively increasing numbers? Note that the sequence 1, 2, 3 is consecutively increasing, while 1, 4, 6 is not.

Solution:

We solve this problem with complementary counting, a way of counting used when solving problems where the number of allowed cases is much harder to enumerate than the number of disallowed cases. First, the total number of permutations without restrictions are 9!. Now we need to subtract from that number the permutations which give at least 7 consecutively increasing numbers. Note that any sequence of consecutive numbers can only begin from the first, second or third position. Therefore, there are three sets we need to consider:

C_1 = The set of sequences with at least 7 consecutive numbers beginning from the first position in line.

C_2 = The set of sequences with at least 7 consecutive numbers beginning from the second position in line.

C_3 = The set of sequences with at least 7 consecutive numbers beginning from the third position in line.

The number of sequences that cannot be included is then represented by $|C_1 \cup C_2 \cup C_3|$. By the principle of inclusion-exclusion (PIE), we can compute this as

$$|C_1 \cup C_2 \cup C_3| = |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_1 \cap C_3| - |C_2 \cap C_3| + |C_1 \cap C_2 \cap C_3|$$

The cardinality $|C_1|$ can be computed using the multiplication rule:

> Step 1: Select the number at the first position: The only options for the first number are 1, 2, and 3, so this can be done in 3 ways.
>
> Step 2: Select the next six numbers: Since the numbers are consecutively increasing, this can be done in 1 way.
>
> Step 3: Select the eighth number: There are two numbers available at this point, so it can be done in 2 ways.
>
> Step 4: Select the last number: With only one number remaining, this can be done in 1 way.

Therefore, $|C_1| = 3 \times 1 \times 2 \times 1 = 6$.

Similarly, the cardinality of C_2 and C_3 are also 6.

Now, we must calculate the cardinalities of the intersections of sets:

$|C_1 \cap C_2|$ can also be calculated using the multiplication rule:

> Step 1: Pick the first number. This can be done in two ways, as the first number can only be 1 or 2.
>
> Step 2: Pick the next 7 numbers. Because the sequence is consecutively increasing from 1-7 and from 2-8, there is only one way to do this.

Step 3: Pick the last number. There is only one number remaining, so this can be done in one way.

Thus, $|C_1 \cap C_2| = 2$. Similarly, we have $|C_2 \cap C_3| = 2$.

For $|C_1 \cap C_3|$, note that this can only be the sequence $1, 2, 3, \ldots, 9$. The same is true for $|C_1 \cap C_2 \cap C_3|$.

Putting this all together, we get the following expression:

$$\begin{aligned}|C_1 \cup C_2 \cup C_3| &= |C_1| + |C_2| + |C_3| - |C_1 \cap C_2| - |C_1 \cap C_3| - |C_2 \cap C_3| + |C_1 \cap C_2 \cap C_3| \\ &= 6 + 6 + 6 - 2 - 2 - 1 + 1 \\ &= 14\end{aligned}$$

Finally we can deduct this number from the total number of permutations without restrictions, making the answer

$$\boxed{9! - 14}$$

Problem 3:

Prove that the product of a non-zero rational and irrational number is irrational.

Solution:

Let us first rewrite the claim as: if a is a non-zero rational number and b is an irrational number, then their product ab is irrational.

We prove the claim using contradiction. In order to do a proof by contradiction, we need to first assume the negation of our statement and then arrive at a false, or contradictory, statement. **Assume for contradiction that a is a non-zero rational number and b is an irrational number, and their product ab is rational.**

Rational numbers can be written as a fraction of two integers where the denominator is non-zero. Since a is a non-zero rational number, we can rewrite it as $a = \frac{p}{q}$ where $p, q \in \mathbb{Z}$ and $p, q \neq 0$.

Let $c = ab$. Since c is a rational number, we can rewrite it as $c = \frac{y}{z}$ where $y, z \in \mathbb{Z}$ and $z \neq 0$. Plugging these in, we get that,

$$c = ab \longrightarrow \frac{y}{z} = \frac{pb}{q}$$

Rearranging the equation and solving for b gives us,

$$b = \frac{yq}{zp}$$

Since yq and zp are integers and $zp \neq 0$, then by definition, b is rational. This is a contradiction (since we assumed that b is an irrational number), and this proves our initial statement.