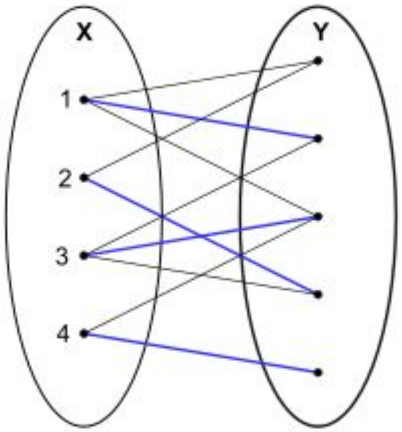
# CIS 160 Recitation #12

Matchings, Relations, Equivalence Classes

# Matching in Bipartite Graphs

**Example. [Hall's Theorem]** Let $G = (X, Y, E)$ be a bipartite graph. For any set $S$ of vertices, let $N_G(S)$ be the set of vertices adjacent to vertices in $S$. Prove that $G$ contains a matching that saturates every vertex in $X$ iff $|N_G(S)| \geq |S|, \forall S \subseteq X$. The condition "For all $S \subseteq X, |N(S)| \geq |S|$" is called Hall's condition.



# Relations

A **binary relation** is a set of ordered pairs.

A binary relation R from a set A to a set B is a subset of the cartesian product A x B.

Let $A = \{1, 2, 3, 4\}$ and $B = \{a, b, c\}$. Consider the following relations.

$$
\begin{aligned}
R_1 &= \{(1,1), (1,2), (2,2), (2,3)\} \\
R_2 &= \{(1,2), (2,3), (3,4), (4,1), (4,4)\} \\
R_3 &= \{(1,a), (2,a), (3,b), (4,c)\} \\
R_4 &= \{(a,1), (a,3), (a,4), (c,1)\} \\
R_5 &= \{(a,a), (a,b), (1,c)\}
\end{aligned}
$$

# Properties of Relations

Let $R$ be a relation defined on set $A$. We say that $R$ is

- *reflexive*, if for all $x \in A$, $(x, x) \in R$.
- *irreflexive*, if for all $x \in A$, $(x, x) \notin R$.
- *symmetric*, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- *antisymmetric*, if for all $x, y \in A$, $x \, R \, y$ and $y \, R \, x \implies x = y$.
- *transitive*, if for all $x, y, z \in A$, $x \, R \, y$ and $y \, R \, z \implies x \, R \, z$.

A relation R on a set A is an **equivalence relation** if and only if it is reflexive, symmetric, and transitive.

# Equivalence Classes

Let R be an equivalence relation on a set A and let a be an element of A.

The **equivalence class** of a, denoted by $[a]_R$, is the set of all elements of A related to a by R.

$$[a]_R = \{x \in A \mid a \, R \, x\}$$

If b is in the equivalence class of a we call b a **representative** of this equivalence class.

# Equivalence Classes (cont.)

Let R be an equivalence relation on a set A and let a and b be elements of A.

Then, the following statements are equivalent:

(i) $b \in [a]$  (ii) $[a] = [b]$  (iii) $[a] \cap [b] \neq \emptyset$