CIS 160

Recitation Guide - Week 11

Topics Covered: Relations, Probability Distributions, Probabilistic Method

Problem 1:

Consider a set A with $n \geq 1$ elements. We color independently each of the elements of A red with probability $\frac{1}{3}$ and blue with probability $\frac{2}{3}$. Let R be the "is the same color as" relation on A, ie. if a is the same color as b, then $(a, b) \in R$.

a) Is R an equivalence relation? If so, what are its equivalence classes?

b) Calculate the expected value of $|R|$.

Solution:

a) R is an equivalence relation:

- Reflexive: $\forall a \in A$, a must be the same color as itself, so aRa.
- Symmetric: given aRb, then a must be the same color as b, so b and a must be both blue or both red. Therefore b must be the same color as a, so bRa.
- Transitive: given aRb and bRc, then a must be the same color as b, and b must be the same color as c. Since b only has one color, then a and c must be the same color.

Since every element in an equivalence class defined by R must be related by R, then each element in such an equivalence class has the same color. Therefore, R determines two equivalence classes of A: in one equivalence class we have all the blue elements and in the other equivalence class we have all the red elements. These determine a partition of A based on color.

b) The elements of R are ordered pairs (x, y) where $x, y \in A$. The sample space is all the possible cardinalities of R.

First we examine the case $x \neq y$. For the pair (x, y) to be in R both x and y must be colored with the same color. Using the independence, the probability that they are both red is $(\frac{1}{3})(\frac{1}{3}) = \frac{1}{9}$. Similarly, the probability that they are both blue is $(\frac{2}{3})(\frac{2}{3}) = \frac{4}{9}$. Since these are disjoint, the probability that they are colored with the same color is $(1/9) + (4/9) = 5/9$, so the probability that $(x, y) \in R$ is $5/9$. Note that there are $n(n-1)$ such (x, y) where $x \neq y$.

Next we examine the case $x = y$. In this case, $(x, y) \in R$ must be true, by reflexivity, , so the probability that $(x, y) \in R$ is 1. Note that there are n such (x, y) where $x = y$.

Now define for each $(x, y) \in A \times A$ an indicator random variable $X_{x,y}$ that is 1 when $(x, y) \in R$ and 0 otherwise. We have $\mathbf{E}(X_{x,y}) = \Pr[(x, y) \in R]$ which equals $5/9$ when $x \neq y$ and 1 when

$x = y$. By linearity of expectation, where we let the sum range over all $(x, y) \in A \times A$:

$$\begin{aligned}
\mathbf{E}\Big[|R|\Big] &= \mathbf{E}\left[\sum_{x,y} X_{x,y}\right] \\
&= \sum_{x,y} \mathbf{E}[X_{x,y}] \\
&= \sum_{x \neq y} \mathbf{E}[X_{x,y}] + \sum_{x} \mathbf{E}[X_{x,x}] \\
&= \frac{5}{9} \cdot n(n-1) + 1 \cdot n \\
&= \frac{n(5n+4)}{9}
\end{aligned}$$

Problem 2:
You are at an auction for a box of money. The amount of money in the box is unknown to you, and is secretly determined by Tien. Tien flips a biased coin 100 times, with a 1/3 chance of getting heads, and for each heads that appears, he puts a dollar in the box.

There is only one other bidder at the auction, Elyssa, who rolls a 6-sided fair die until she gets a 6, and for each roll adds $5 to her bid.

Everyone who attends the auction reveals their bid at the same time, and the person with the highest bid pays that amount of money to get the box. (Assume you can only bet in whole dollar amounts.)

a) Let's say you want to bid strictly more than the expected value of Elyssa's bid (so you win the box), but strictly less than the expected value of the box (so you still make money). Is that possible?

b) What if Elyssa bids according to a 7-sided die, rolling until she gets a 7? In expectation and using the same strategy as a), can you still make money?

Solution:

Note that the sample space is the set of all ordered pairs (x, y) where x represents the sequence of 100 biased coin flips and y represents the sequence of 6-sided die rolls ending with the first 6. That is, $\Omega = \Omega_M \times \Omega_B$, where Ω_M is the sample space for the biased coin flips and Ω_B is the sample space for Elyssa's die rolls.

a) Let M be the amount of money in the box, and let B be the amount that Elyssa bids. We need to calculate the **E**[M] and **E**[B] bid.

Note that M is a binomial random variable with $n = 100$ and $p = \frac{1}{3}$. Therefore:

$$\begin{aligned}\mathbf{E}[M] &= np \\ &= \frac{100}{3} \approx 33.33\end{aligned}$$

Let $B = 5B_6$, where B_6 is the random variable denoting the number of rolls of the 6-sided die up to and including the first roll of 6. Note that B_6 is a geometric random variable with $p = \frac{1}{6}$. By the Linearity of Expectation:

$$\begin{aligned}\mathbf{E}[B] &= \mathbf{E}[5B_6] \\ &= 5\mathbf{E}[B_6] \\ &= 5 \cdot \frac{1}{p} \\ &= 5 \cdot 6 = 30\end{aligned}$$

Since $30 < 33.33$ and there is a whole dollar amount between the two, we can bid an amount between \$30 and \$33.33 (exclusive) and expect to make money.

b) $\mathbf{E}[M]$ is the same as before, but $\mathbf{E}[B]$ will be different (and the sample space should y represents the sequence of 7-sided die rolls ending with the first 7).

Now $B = 5B_7$, where B_7 is the random variable denoting the number of rolls of the 7-sided die up to and including the first roll of 7. Note that B_7 is a geometric random variable with

$p = \frac{1}{7}$. By the Linearity of Expectation:

$$\begin{aligned}\mathbf{E}[B] &= \mathbf{E}[5B_7] \\ &= 5\mathbf{E}[B_7] \\ &= 5 \cdot \frac{1}{p} \\ &= 5 \cdot 7 = 35\end{aligned}$$

Since $35 > 33.33$, then there is no amount of money we can bid where we can expect to make money according to this strategy.

Problem 3:

Consider an undirected graph $G = (V, E)$, where $|V| = n$ and $|E| = m$. Prove that there is a partition of V into two disjoint sets A and B such that at least $\frac{m}{2}$ edges connect vertices across A and B.

Solution:

We approach this using the probabilistic method: we construct A and B randomly by assigning each vertex to either side of the partition with equal probability. We define a random variable X that represents the total number of edges across this cut.

Let the indicator random variables X_i be 1 if the ith edge $e = \{u, v\}$ connects a vertex in A to a vertex in B, and 0 otherwise. Further, note that the probability that an edge connects a vertex in A to a vertex in B is $\frac{1}{2}$ (since there are two out of the four possible assignments of u and v to sets A and B which cause edge e to cross the cut from A to B), i.e. $\Pr[X_i = 1] = \frac{1}{2}$, which means that $\mathbb{E}[X_i] = \frac{1}{2}$.

Using the linearity of expectation:

$$\mathbb{E}[X] = \sum_{i=1}^{m} \mathbb{E}[X_i] = \sum_{i=1}^{m} \Pr[X_i = 1] = \frac{m}{2}$$

Given that the expected value of our random variable X is $\frac{m}{2}$, X must take on a value greater than or equal to $\frac{m}{2}$ with non-zero probability; that is, there is at least one outcome in the sample space, corresponding to a partition A and B, such that the number of edges crossing the cut is at least $\frac{m}{2}$.