CIS 160 Recitation 10

Relations

November 18-19, 2021

Relations and Probabilistic Methods

- A binary relation is a set of ordered pairs.
- For example, $R = \{(1,2),(2,3),(5,4)\}$
- $(1,2) \in R$: 1 is related to 2 by relation R, we denote this by $1R2$.
- A binary relation R from set A to B is a subset of the Cartesian product $A \times B$.

Properties of Relation

- *Reflexive:* for all $x \in A, (x, x) \in R$.
- *Irreflexive:* for all $x \in A, (x, x) \notin R$.
- *Symmetric:* for all $x, y \in A, (x, y) \in R \implies (y, x) \in R$.
- *Antisymmetric:* for all $x, y \in A, (x, y) \in R$ and $(y, x) \in R \implies x = y$.
- *Transitive:* for all $x, y, z \in A, (x, y) \in R$ and $(y, z) \in R \implies (x, z) \in R$.
- *Symmetric* and *antisymmetric* are not opposites.

Equivalence Relations and Classes

- A relation R on a set A is an **equivalence relation** if and only if it is reflexive, symmetric, and transitive.
- **The equivalence class** of $a \in A$, denoted by $[a]_R$, is the set of all elements of A related by R to a.

$$[a]_R = \{x \in A \mid aRx\}$$

- If $b \in [a]_R$, then b is called the *representative* of this equivalence class.
- Let R be an equivalence relation on set A. The following statements for $a, b \in A$ are equivalent: (i) $b \in [a]_R$, (ii) $[a]_R = [b]_R$, and (iii) $[a]_R \cap [b]_R \neq \emptyset$.

Relations: Operations, Inverse, Composite

- Since a relation R from set A to set B is a subset of $A \times B$, operations that apply to sets apply to relations, such as intersection and union
- **Inverse relation:** $R^{-1} = \{(b, a) \mid (a, b) \in R\}$
- **Composite of relations:** Let R be a relation from A to B and S be a relation from B to C. The composition of S with R is a relation from A to C:

  $$S \circ R = \{(x, z) \mid \text{there exists a } y \in B \text{ such that } xRy \text{ and } ySz\}$$

The Probabilistic Method

The *probabilistic method* is a proof technique for proving "there exists an x such that $P(x)$ is true" claims without directly providing x.

The general procedure of a probabilistic method proof is:

- Construct a random procedure to generate candidates for x.
- Let x be the candidate generated by the random process. Show that $\Pr[P(x)] > 0$, or if $P(x)$ is phrased numerically you can show that in expectation $P(x)$ is true.
- Conclude that there must be some x, corresponding to an outcome in the sample space, such that $P(x)$ is true.