CIS 160

# Recitation Guide - Week 10

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**Topics Covered:** Eulerian Graphs, Linearity of Expectation, Graphs

**Problem 1:**

Given a directed graph $G = (V, E)$, prove that the sum of the outdegrees of all the nodes in a directed graph is equal to the number of edges.

**Solution:**

Note that in a directed graph, since each edge $e = (u, v)$ is a directed edge from $u$ to $v$, then each edge contributes exactly one outdegree (and one indegree). Therefore the sum of the outdegrees equals the number of edges.

**Problem 2:**
For any simple, undirected graph $G = (V, E)$, for any distinct $u, v \in V$, prove that if there is a walk from $u$ to $v$ then there is a path from $u$ to $v$.

**Solution:**

Assume, for the purpose of contradiction, a walk from distinct vertices $u$ to $v$ exists in $G$, but there is no path from $u$ to $v$. If there is a walk from $u$ to $v$, then there must be some shortest walk $w$ from $u$ to $v$. Since there are no paths from $u$ to $v$, then $w$ is not a path, so $w$ must have repeated vertices. Let $x$ be the first repeated vertex in $w$ going from $u$ to $v$. Therefore, $w$ is the form:

$$u \rightsquigarrow x \rightsquigarrow x \rightsquigarrow v$$

We can create a smaller walk $w'$ by removing the vertices and edges of $w$ after the first occurrence of $x$, up to and including the second occurrence of $x$. That is, $w'$ omits the $x \rightsquigarrow x$ portion of $w$, but keeping the first occurrence of $x$. Since we have removed at least one edge from $w$ to get $w'$, the length of $w'$ is strictly less than the length of $w$, which contradicts our assumption that $w$ is the shortest walk from $u$ to $v$.

Therefore when there is a walk between two vertices in a simple undirected graph, there is a path.

**Problem 3:**
Oliver, as a busy college student, hasn't done laundry in weeks. In particular, he realizes that he has no more socks to wear, so he goes to the laundry room and throws in his 2 distinct pairs of socks. However, the machine is broken, and so it only returns 2 of his socks at random! Note that the two socks in a pair are also distinguishable.

(a) What is the expected number of pairs that he can wear now?

(b) What if he throws in $n$ pairs and only gets $k$ $(k > 1)$ socks back?

**Solution:**

(a) We define the sample space $\Omega = \{\{s_1, s_2\} \mid s_i \text{ is a sock}\}$. Notice that the sample space is uniform. We will denote the number of pairs Oliver can wear using random variable $S$. Since only 2 socks are returned, $S$ can only take on values 0 and 1. Then we have

$$Pr[S = 0] = \frac{\binom{2}{1} \cdot \binom{2}{1}}{\binom{4}{2}} = \frac{2}{3}$$

$$Pr[S = 1] = \frac{2 \cdot \binom{2}{2}}{\binom{4}{2}} = \frac{1}{3}$$

Hence,

$$E[S] = 0 \cdot Pr[S = 0] + 1 \cdot Pr[S = 1] = \frac{1}{3}$$

(b) We define the sample space $\Omega = \{\{s_1, s_2, \ldots, s_k\} \mid s_i \text{ is a sock}\}$. Notice that the sample space is uniform, since each subset of $k$ socks are equally likely to be returned. If Oliver throws in $n$ pairs, let $I_1, I_2, \ldots, I_n$ be indicator random variables where $I_i = 1$ if the pair $i$ is returned to him (and $I_i = 0$ if the pair is not). Now, we want to find $E[S]$. Notice that

$$\begin{aligned} E[S] &= E\left[\sum_{i=1}^{n} I_i\right] \\ &= \sum_{i=1}^{n} E[I_i] \qquad \text{by Linearity of Expectation} \end{aligned}$$

For an arbitrary $I_i$, notice that $E[I_i] = \Pr[I_i = 1] = \Pr[E_i]$, where $E_i$ is the event that the $i^{th}$ pair was returned. Since the sample space is uniform, we have that $\Pr[E_i] = \frac{|E_i|}{\binom{2n}{k}}$. We also know that $|E_i| = \binom{2n-2}{k-2}$, the number of ways of returning $k$ socks that includes pair $i$. Thus, we have:

$$\begin{aligned} E[S] &= \sum_{i=1}^{n} E[I_i] \\ &= n \cdot \frac{\binom{2n-2}{k-2}}{\binom{2n}{k}} \end{aligned}$$

**Problem 4:**
Prove that if a graph has an Eulerian circuit, its edges can be partitioned into a set of edge-disjoint cycles (that is, cycles that do not share any edges).

**Solution:**

We will prove this by induction on the number of edges $m$.

Base Case: $m = 3$. For a graph with 3 edges to have an Eulerian circuit, it must be the cycle graph on 3 vertices. In this case, our partition of the edges would just have one set- the entire set of edges, which gives one cycle. Therefore, we can partition our edge set into a set of edge-disjoint cycles. ✓

(Why can't the base case be a number of edges less than 3?)

Induction Hypothesis: Assume the claim holds for all $3 \leq j \leq k$, where $j, k \in \mathbb{N}$. That is, assume that for a graph with $j$ edges, if that graph has an Eulerian circuit, then its edges can be partitioned into a set of edge-disjoint cycles.

Induction Step: We now consider an arbitrary graph $G$ with $k + 1$ edges that contains an Eulerian circuit. Let us consider the Eulerian circuit in $G$. It will be of the form $v_1 v_2 v_3 ... v_m$. We know that since this is an Eulerian circuit, we must eventually visit a vertex we have seen earlier (at the least, $v_1$ must equal $v_m$ for this to even be a circuit).

Intuitively, we want to remove the edges of a cycle so that we can apply our induction hypothesis. To show that such a cycle exists, let us consider the smallest circuit in the Eulerian circuit. This will be a cycle.

(i.e. we consider a pair of vertices on this circuit $v_i$ and $v_j$, $i < j$, such that $v_i = v_j$. More specifically, let us consider the pair of $v_i$ and $v_j$ where $j - i$ is minimized. We know that since $v_i = v_j$ and we can follow the circuit from $v_i$ back to itself (now enumerated $v_j$), we have found a circuit. Moreover, since $j - i$ is minimized, we know that this circuit cannot contain a smaller circuit; therefore, it must be a cycle.)

Alternatively, we can use the same process from lecture 10T to find a cycle to remove. That is, since $G$ has at least 3 edges, is connected, and every vertex of $G$ has even degree, then $\delta(G) \geq 2$, which means $G$ contains a cycle.

Let us remove all the edges in this cycle that we have found, and add them to a set $S$ for now. We now have a graph $G'$, where $E(G') = E(G) \setminus S$. $G'$ may be disconnected, and we cannot say for sure that it has an Eulerian circuit. However, we know one thing. We have only removed the edges in our cycle that we found earlier. For each of the vertices in that cycle, we took away 2 from each of their degrees. For all other vertices in the graph, we did not change their degree. Since we originally had an Eulerian circuit, all vertices in $G$ must have had even degree, and in $G'$, they must still have even degree.

For each connected component of $G'$, then, it must either be a lone vertex or contain an Eulerian circuit (because all vertices have even degree within the CC). Furthermore, each of these connected components cannot possibly have more than $k$ edges. Moreover, for the CCs that are not lone vertices, there must be at least 3 edges. If there was one edge, then there can only be two vertices each with degree 1. If there were two edges, then it must be a path graph of length 3, and the two endpoints will have degree 1. In both of these cases, not all vertices will have even degree. Hence, there must be at least 3 edges in CCs that are not lone vertices. Therefore, by our IH, each connected component's (not the lone vertex CCs) edges can be partitioned into a set of edge-disjoint cycles. Between partitions for each CC, there cannot be any shared edges (that would contradict that they are separate CCs). Therefore, we can combine each CC's set of edge-disjoint cycles with S to construct a partition on $G$'s $k+1$ edges that forms a set of edge-disjoint cycles. This completes the IS.