CIS 160

Recitation Guide - Week 10

Topics Covered: Tail Bounds, Total Expectation, Memoryless Property, Hall's Theorem

Problem 1:

Recall the following problem from last week's recitation:

A 10 digit number with no zeroes is chosen by independently and randomly selecting each digit (1 - 9). Let N be the number of digits missing from the 10 digit number. Last week, we calculated $\mathbb{E}[N] \approx 2.772$ and $\text{Var}[N] \approx 0.9232$.

a) Using Markov's Inequality, what is the lower bound of the probability that less than 6 digits are missing?

b) How can you improve the bound you obtained above?

Solution:

a) We are looking to lower-bound $\Pr[N < 6]$. Note that Markov's Inequality gives information about upper bounds on the probability that N is large. However, we also know that $\Pr[N < 6] = 1 - \Pr[N \geq 6]$. Also, keep in mind we can apply Markov's Inequality because N represents the number of missing digits, so N is a non-negative random variable. We begin from the information that Markov's Inequality guarantees us:

$$\Pr[N \geq a] \leq \frac{\mathbb{E}[N]}{a} \qquad \text{(Markov's Inequality)}$$

$$\Pr[N \geq 6] \leq \frac{\mathbb{E}[N]}{6} \qquad (a = 6)$$

$$\Pr[N \geq 6] \leq \frac{2.772}{6} \qquad (\mathbb{E}[N] \approx 2.772)$$

Solving for lower bound,

$$\begin{aligned} \Pr[N < 6] &= 1 - \Pr[N \geq 6] \\ &\geq 1 - \frac{2.772}{6} \\ &= 0.5381 \end{aligned}$$

b) We can use Chebyshev's inequality:

$$\Pr[|N - \mathbb{E}[N]| \geq a] \leq \frac{\text{Var}[N]}{a^2} \qquad \text{(Chebyshev's Inequality)}$$

Choose $a = 6 - 2.772 = 3.228$. We have

$$\Pr[|N - 2.772| \geq 3.228] \leq \frac{0.9232}{3.228^2} \approx 0.0886$$

As N is non-negative, we have that

$$\Pr[N \geq 6] \leq 0.0886$$

Using the same rearranging of terms from part a), we get

$$\Pr[N < 6] \geq 0.9114$$

Problem 2:

For a geometric random variable X with parameter p, where $n > 0$ and $k \geq 0$, we have the memoryless property

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

The following is the definition of conditional expectation.

$$\mathbb{E}[Y \mid Z = z] = \sum_y y \cdot \Pr[Y = y \mid Z = z],$$

a) Prove the law of total expectation below. Given any random variables X, Y, defined in the same sample space,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X|Y = y] \Pr[Y = y]$$

b) Calculate the expectation of a geometric random variable with the memoryless property and the law of total expectation.

Solution:

a) We have

$$\begin{aligned}
\mathbb{E}[X] &= \sum_x x \cdot \Pr[X = x] \\
&= \sum_x x \cdot \sum_y \Pr[X = x|Y = y] \cdot \Pr[Y = y] && \text{(By Law of Total Probability)} \\
&= \sum_y \Pr[Y = y] \cdot \sum_x x \cdot \Pr[X = x|Y = y] \\
&= \sum_y \Pr[Y = y] \cdot \mathbb{E}[X|Y = y]
\end{aligned}$$

b) We calculate the expectation of a geometric random variable X with parameter p as follows. Seeing as we have the memoryless property, we condition X on the result of the first trial.

Formally, let Y be the indicator random variable that represents the outcome of the first Bernoulli trial, where $Y = 0$ if the first trial is a failure and $Y = 1$ otherwise. Using the law of total expectation, we have

$$\begin{aligned}
\mathbb{E}[X] &= \mathbb{E}[X \mid Y = 0] \cdot \Pr[Y = 0] + \mathbb{E}[X \mid Y = 1] \cdot \Pr[Y = 1] \\
&= \mathbb{E}[X \mid Y = 0] \cdot \Pr[Y = 0] + 1 \cdot p
\end{aligned}$$

We see that if the first trial was a success, then expectation of X is 1, as there will be no more trials.

Intuitively, since X is memoryless, if the first trial was a failure, the expected number of trials would just be $\mathbb{E}[X] + 1$. Rigorously, we attempt to use the memoryless property on the first term. We have

$$\begin{aligned}
\mathbb{E}[X \mid Y = 0] &= \sum_{x=1}^{\infty} x \cdot \Pr[X = x \mid Y = 0] \\
&= 1 \cdot \Pr[X = 1 \mid Y = 0] + \sum_{x=2}^{\infty} x \cdot \Pr[X = x \mid Y = 0] && \text{(Splitting the sum)}
\end{aligned}$$

Note that if $Y = 0$, then the first trial is a failure. Then X cannot equal 1, because $X = 1$ means that there was a success on the first trial. Therefore $\Pr[X = 1 \mid Y = 0] = 0$.

Note also that $Y = 0$ if and only if $X > 1$, since we must have gone through more than one trial to obtain a success. Substituting these in, we get:

$$\begin{aligned}
\mathbb{E}[X \mid Y = 0] &= 0 + \sum_{x=2}^{\infty} x \cdot \Pr[X = x \mid X > 1] \\
&= \sum_{x=2}^{\infty} x \cdot \Pr[X = (x-1) + 1 \mid X > 1] \\
&= \sum_{x=2}^{\infty} x \cdot \Pr[X = x - 1] && \text{(By the memoryless property)} \\
&= \sum_{x=1}^{\infty} (x+1) \cdot \Pr[X = x] && \text{(Shifting the lower bound back to 1)} \\
&= \sum_{x=1}^{\infty} x \cdot \Pr[X = x] + \sum_{x=1}^{\infty} \Pr[X = x] \\
&= \mathbb{E}[X] + 1
\end{aligned}$$

Hence, putting everything together, we have

$$\begin{aligned}
\mathbb{E}[X] &= (1-p) \cdot (\mathbb{E}[X] + 1) + p \\
\mathbb{E}[X] &= (1-p) \cdot \mathbb{E}[X] + (1-p) \cdot 1 + p \\
\mathbb{E}[X] - (1-p) \cdot \mathbb{E}[X] &= 1 - p + p \\
\mathbb{E}[X] \cdot [1 - (1-p)] &= 1 \\
\mathbb{E}[X] \cdot (p) &= 1 \\
\mathbb{E}[X] &= \frac{1}{p}
\end{aligned}$$

Problem 3:
Consider a normal chessboard (an 8×8 grid). In each row and in each column there are exactly n pieces, where $0 < n \leq 8$. Prove that we can pick 8 pieces such that no two of them are in the same row or column.

Solution:

We construct a bipartite graph G as follows. Let X be the set of rows modeled as vertices. Let Y be the set of columns modeled as vertices. Let E be the set of edges such that if a piece exists in row i and column j, then there is an edge between $x_i \in X$ and $y_j \in Y$. Note that the graph must be bipartite because no edges exist between two vertices in X or two vertices in Y.

The question asks us to find a matching: can we match each of the 8 rows to a unique column? Note that this would mean that we could pick 8 edges (in our matching) that are not in the same row or same column.

We must prove the existence of such a perfect matching. First, note that the size of our two bipartite sets X and Y are the same since there are exactly 8 rows and 8 columns; in other words, $|X| = |Y| = 8$. Hence, if we can find a matching that saturates X, then it must also saturate Y (and so is a perfect matching). To prove the existence of this matching, we show that Hall's Condition is satisfied, that is that $|N_G(S)| \geq |S|, \forall S \subseteq X$.

Consider an arbitrary but particular subset $A \subseteq X$ (of the rows). Recall that there are n pieces in each row and n pieces in each column. Thus, there must be $n|A|$ edges from A to $N_G(A)$. We also know that each column in $N_G(A)$ has at most n edges back to A, meaning that there are at most $n|N_G(A)|$ edges from $N_G(A)$ to A. This means that $n|A| \leq n|N_G(A)|$, meaning that $|A| \leq |N_G(A)|$. This satisfies Hall's Condition, leading us to prove the existence of our matching.