CIS 160

Recitation Guide - Week 10

Topics Covered: Tail Bounds, Total Expectation, Memoryless Property, Hall's Theorem

Problem 1:

Recall the following problem from last week's recitation:

A 10 digit number with no zeroes is chosen by independently and randomly selecting each digit (1 - 9). Let N be the number of digits missing from the 10 digit number. Last week, we calculated $\mathbb{E}[N] \approx 2.772$ and $\text{Var}[N] \approx 0.9232$.

a) Using Markov's Inequality, what is the lower bound of the probability that less than 6 digits are missing?

b) How can you improve the bound you obtained above?

Problem 2:

For a geometric random variable X with parameter p, where $n > 0$ and $k \geq 0$, we have the memoryless property

$$\Pr[X = n + k \mid X > k] = \Pr[X = n]$$

The following is the definition of conditional expectation.

$$\mathbb{E}[Y \mid Z = z] = \sum_y y \cdot \Pr[Y = y \mid Z = z],$$

a) Prove the law of total expectation below. Given any random variables X, Y, defined in the same sample space,

$$\mathbb{E}[X] = \sum_y \mathbb{E}[X | Y = y] \Pr[Y = y]$$

b) Calculate the expectation of a geometric random variable with the memoryless property and the law of total expectation.

Problem 3:
Consider a normal chessboard (an 8×8 grid). In each row and in each column there are exactly n pieces, where $0 < n \leq 8$. Prove that we can pick 8 pieces such that no two of them are in the same row or column.