

# Mathematical Foundations of Computer Science

## Lecture Outline

October 22, 2020

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**Example.** An urn contains 5 white and 10 black balls. A fair die is rolled and that number of balls are chosen from the urn.

- (a) What is the probability that all of the balls selected are white?
- (b) What is the conditional probability that the die landed on 3 if all the balls selected are white?

**Solution.** We will consider the following events.

$W$ : event that all of the balls chosen are white.

$D_i$ : event that the die landed on  $i$ ,  $1 \leq i \leq 6$ .

- (a) We want to find  $\Pr[W]$ . We can do this as follows.

$$\begin{aligned}\Pr[W] &= \sum_{i=1}^6 \Pr[W \cap D_i] \\ &= \sum_{i=1}^6 \Pr[D_i] \Pr[W|D_i] \\ &= \sum_{i=1}^6 \frac{1}{6} \frac{\binom{5}{i}}{\binom{15}{i}} \\ &= \frac{1}{6} \left( \frac{5}{15} + \frac{10}{105} + \frac{10}{455} + \frac{5}{1365} + \frac{1}{3003} \right) \\ &= 0.075\end{aligned}$$

- (b) We want to find  $\Pr[D_3|W]$ . This can be done as follows.

$$\begin{aligned}\Pr[D_3|W] &= \frac{\Pr[D_3 \cap W]}{\Pr[W]} \\ &= \frac{\Pr[D_3] \times \Pr[W|D_3]}{\Pr[W]} \\ &= \frac{1/6 \times \binom{5}{3} / \binom{15}{3}}{0.075} \\ &= \frac{1/6 \times 10/455}{0.075} \\ &= \frac{0.00366}{0.075} \\ &= 0.048\end{aligned}$$

**Independent Events.** Two events  $A$  and  $B$  are *independent* if and only if  $\Pr[A \cap B] = \Pr[A] \times \Pr[B]$ . This definition also implies that if the conditional probability  $\Pr[A|B]$  exists, then  $A$  and  $B$  are independent events if and only if  $\Pr[A|B] = \Pr[A]$ .

Events  $A_1, A_2, \dots, A_n$  are *mutually independent* if  $\forall i, 1 \leq i \leq n$   $A_i$  does not “depend” on any combination of the other events. More formally, for every subset  $I \subseteq \{1, 2, \dots, n\}$ ,

$$\Pr[\bigcap_{i \in I} A_i] = \prod_{i \in I} \Pr[A_i]$$

In other words, to show that  $A_1, A_2, \dots, A_n$  are mutually independent we must show that all of the following hold.

$$\begin{aligned} \Pr[A_i \cap A_j] &= \Pr[A_i] \cdot \Pr[A_j] \quad \forall \text{ distinct } i, j \\ \Pr[A_i \cap A_j \cap A_k] &= \Pr[A_i] \cdot \Pr[A_j] \cdot \Pr[A_k] \quad \forall \text{ distinct } i, j, k \\ \Pr[A_i \cap A_j \cap A_k \cap A_l] &= \Pr[A_i] \cdot \Pr[A_j] \cdot \Pr[A_k] \cdot \Pr[A_l] \quad \forall \text{ distinct } i, j, k, l \\ &\dots \\ \Pr[A_1 \cap A_2 \cap \dots \cap A_n] &= \Pr[A_1] \Pr[A_2] \dots \Pr[A_n] \end{aligned}$$

The above definition implies that if  $A_1, A_2, \dots, A_n$  are mutually independent events then

$$\Pr[A_1 \cap A_2 \cap \dots \cap A_n] = \Pr[A_1] \times \Pr[A_2] \times \dots \times \Pr[A_n]$$

However, note that  $\Pr[A_1 \cap A_2 \cap \dots \cap A_n] = \Pr[A_1] \times \Pr[A_2] \times \dots \times \Pr[A_n]$  is not a sufficient condition for  $A_1, A_2, \dots, A_n$  to be mutually independent.

Do not confuse the concept of disjoint events and independent events. If two events  $A$  and  $B$  are disjoint and have a non-zero probability of happening then given that one event happens reduces the chances of the other event happening to zero, i.e.,  $\Pr[A|B] = 0 \neq \Pr[A]$ . Thus by definition of independence, events  $A$  and  $B$  are not independent.

**Example.** Two cards are sequentially drawn (without replacement) from a well-shuffled deck of 52 cards. Let  $A$  be the event that the two cards drawn have the same value (e.g. both 4s) and let  $B$  be the event that the first card drawn is an ace. Are these events independent?

**Solution.** To decide whether the two events are independent we need to check whether  $\Pr[A \cap B] = \Pr[A] \Pr[B]$ .

$$\begin{aligned} \Pr[A] &= \frac{3}{51} = \frac{1}{17} \\ \Pr[B] &= \frac{4}{52} = \frac{1}{13} \\ \Pr[A \cap B] &= \frac{1}{13} \times \frac{3}{51} \\ &= \frac{1}{221} \\ &= \frac{1}{17} \times \frac{1}{13} \\ &= \Pr[A] \Pr[B] \end{aligned}$$

**Example.** Suppose that a fair coin is tossed twice. Let  $A$  be the event that a head is obtained on the first toss,  $B$  be the event that a head is obtained on the second toss, and  $C$  be the event that either two heads or two tails are obtained. (a) Are events  $A, B, C$  pairwise independent? (b) Are they mutually independent?

**Solution.** Note that  $\Omega = \{HH, HT, TH, TT\}$ .  $A = \{HH, HT\}$ ,  $B = \{HH, TH\}$ ,  $C = \{HH, TT\}$ ,  $A \cap B = \{HH\}$ ,  $A \cap C = \{HH\}$ ,  $B \cap C = \{HH\}$ ,  $A \cap B \cap C = \{HH\}$ . The probabilities of the relevant events are as follows.

$$\begin{aligned} \Pr[A] &= 1/2 \\ \Pr[B] &= 1/2 \\ \Pr[C] &= 1/2 \\ \Pr[A \cap B] &= 1/4 = \Pr[A] \Pr[B] \\ \Pr[A \cap C] &= 1/4 = \Pr[A] \Pr[C] \\ \Pr[B \cap C] &= 1/4 = \Pr[B] \Pr[C] \\ \Pr[A \cap B \cap C] &= 1/4 \neq \Pr[A] \Pr[B] \Pr[C] \end{aligned}$$

Thus we see that  $A, B, C$  are pairwise independent but not mutually independent.

**Example.** Consider the experiment in which we roll a dice twice. Consider the following events.

- $A$ : event that the first roll results in a 1, 2, or a 3.
- $B$ : event that the first roll results in a 3, 4, or a 5.
- $C$ : event that the sum of the two rolls is a 9

Are events  $A, B$ , and  $C$  mutually independent?

**Solution.** We show below that the events are not mutually independent as they are not pairwise independent.

$$\begin{aligned}
 A &= \{(i, j) \mid 1 \leq i \leq 3 \text{ and } 1 \leq j \leq 6\} \\
 B &= \{(i, j) \mid 3 \leq i \leq 5 \text{ and } 1 \leq j \leq 6\} \\
 C &= \{(3, 6), (6, 3), (4, 5), (5, 4)\} \\
 A \cap B &= \{(3, 1), (3, 2), (3, 3), (3, 4), (3, 5), (3, 6)\} \\
 A \cap C &= \{(3, 6)\} \\
 B \cap C &= \{(3, 6), (4, 5), (5, 4)\} \\
 A \cap B \cap C &= \{(3, 6)\} \\
 \Pr[A] &= 1/2 \\
 \Pr[B] &= 1/2 \\
 \Pr[C] &= 1/9 \\
 \Pr[A \cap B \cap C] &= 1/36 = \Pr[A] \cdot \Pr[B] \cdot \Pr[C] \\
 \Pr[A \cap B] &= 1/6 \neq \Pr[A] \cdot \Pr[B] \\
 \Pr[A \cap C] &= 1/36 \neq \Pr[A] \cdot \Pr[C] \\
 \Pr[B \cap C] &= 3/36 \neq \Pr[B] \cdot \Pr[C]
 \end{aligned}$$

## Random Variables

In an experiment we are often interested in some value associated with an outcome as opposed to the actual outcome itself. For example, consider an experiment that involves tossing a coin three times. We may not be interested in the actual head-tail sequence that results but be more interested in the number of heads that occur. These quantities of interest are called *random variables*.

**Definition.** A *random variable*  $X$  on a sample space  $\Omega$  is a real-valued function that assigns to each sample point  $\omega \in \Omega$  a real number  $X(\omega)$ .

In this course we will study discrete random variables which are random variables that take on only a finite or countably infinite number of values.

For a discrete random variable  $X$  and a real value  $a$ , the event “ $X=a$ ” is the set of outcomes in  $\Omega$  for which the random variable assumes the value  $a$ , i.e.,  $X = a \equiv \{\omega \in \Omega \mid X(\omega) = a\}$ . The probability of this event is denoted by

$$\Pr[X = a] = \sum_{\omega \in \Omega: X(\omega)=a} \Pr[\omega]$$

**Definition.** The *distribution* or the *probability mass function* (PMF) of a random variable  $X$  gives the probabilities for the different possible values of  $X$ . Thus, if  $x$  is a value that

$X$  can assume then  $p_X(x)$  is the probability mass of  $X$  and is given by

$$p_X(x) = \Pr[X = x]$$

Observe that  $\sum_x p_X(x) = \sum_x \Pr[X = x] = 1$ . This is because the events  $X = x$  are disjoint and hence partition the sample space  $\Omega$ .

Consider the experiment of tossing three fair coins. Let  $X$  be the random variable that denotes the number of heads that result. The PMF or the distribution of  $X$  is given below.

$$p_X(x) = \begin{cases} 1/8 & \text{if } x = 0 \text{ or } x = 3 \\ 3/8 & \text{otherwise} \end{cases}$$

The definition of independence that we developed for events extends to random variables.

**Definition.** Two random variables  $X$  and  $Y$  are independent if and only if

$$\Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \times \Pr[Y = y]$$

for all values  $x$  and  $y$ . In other words, two random variables  $X$  and  $Y$  are independent if every event determined by  $X$  is independent of every event determined by  $Y$ .

Similarly, random variables  $X_1, X_2, \dots, X_k$  are mutually independent if and only if, for any subset  $I \subseteq [1, k]$  and any values  $x_i, i \in I$ ,

$$\Pr[\cap_{i \in I} X_i = x_i] = \prod_{i \in I} \Pr[X_i = x_i]$$

## Expectation

The PMF of a random variable,  $X$ , provides us with many numbers, the probabilities of all possible values of  $X$ . It would be desirable to summarize this distribution into a representative number that is also easy to compute. This is accomplished by the *expectation* of a random variable which is the weighted average (proportional to the probabilities) of the possible values of  $X$ .

**Definition.** The *expectation* of a discrete random variable  $X$ , denoted by  $\mathbf{E}[X]$ , is given by

$$\mathbf{E}[X] = \sum_i i p_X(i) = \sum_i i \Pr[X = i]$$

Intuitively,  $\mathbf{E}[X]$  is the value we would expect to obtain if we repeated a random experiment several times and took the average of the outcomes of  $X$ .

In our running example, in expectation the number of heads is given by

$$\mathbf{E}[X] = 0 \times \frac{1}{8} + 3 \times \frac{1}{8} + 1 \times \frac{3}{8} + 2 \times \frac{3}{8} = \frac{3}{2}$$

As seen from the example, the expectation of a random variable may not be a valid value of the random variable.

**Example.** When we roll a die what is the result in expectation?

**Solution.** Let  $X$  be the random variable that denotes the result of a single roll of dice. The PMF for  $X$  is given by

$$p_X(x) = \frac{1}{6}, x = 1, 2, 3, 4, 5, 6.$$

The expectation of  $X$  is given by

$$\mathbf{E}[X] = \sum_{x=1}^6 p_x(x) \cdot x = \frac{1}{6} (1 + 2 + 3 + 4 + 5 + 6) = 3.5$$