

Mathematical Foundations of Computer Science

Lecture Outline

October 13, 2020

Review of Definitions: Walk, path, cycle, connected graph, subgraph, induced subgraph, connected component.

Trees

A graph with no cycles is *acyclic*. A *tree* is a connected acyclic graph. A vertex of degree greater than 1 in a tree is called an *internal vertex*, otherwise it is called a *leaf*. A *forest* is an acyclic graph.

Example. Prove that every tree with at least two vertices has at least two leaves and deleting a leaf from an n -vertex tree produces a tree with $n - 1$ vertices.

Solution. A connected graph with at least two vertices has an edge. In an acyclic graph, an endpoint of a maximal non-trivial path (a path that is not contained in a longer path) has no neighbors other than its only neighbor on the path. Hence, the endpoints of such a path are leaves.

Let v be a leaf of a tree T and let $T' = T - v$. A vertex of degree 1 belongs to no path connecting two vertices other than v . Hence, for any two vertices $u, w \in V(T')$, every path from u to w in T is also in T' . Hence T' is connected. Since deleting a vertex cannot create a cycle, T' is also acyclic. Thus, T' is a tree with $n - 1$ vertices.

Example. For a n -vertex graph G , the following are equivalent and characterize trees with n vertices.

- (1) G is a tree.
- (2) G is connected and has exactly $n - 1$ edges.
- (3) G is minimally connected, i.e., G is connected but $G - \{e\}$ is disconnected for every edge $e \in G$.
- (4) G contains no cycle but $G + \{x, y\}$ does, for any two non-adjacent vertices $x, y \in G$.
- (5) Any two vertices of G are linked by a unique path in G .

Solution. (1 \rightarrow 2). We can prove this by induction on n . The property is clearly true for $n = 1$ as G has 0 edges. Assume that any tree with k vertices, for some $k \geq 0$, has $k - 1$ edges. We want to prove that a tree G with $k + 1$ vertices has k edges. From the example we did in last class we know that G has a leaf, say v , and that $G' = G - \{v\}$ is connected. By induction hypothesis, G' has $k - 1$ edges. Since $\deg(v) = 1$, G has k edges.

(2 \rightarrow 3). Note that $G - \{e\}$ has n vertices and $n - 2$ edges. We know that such a graph has at least 2 connected components and hence is disconnected.

(3 \rightarrow 4). We are assuming that removing *any* edge in G disconnects G . If G contains a cycle then removing any edge, say $\{u, v\}$, that is part of the cycle does not disconnect G as any path that uses $\{u, v\}$ can now use the alternate route from u to v on the cycle. Since G is connected there is a path from x to y in G . Let $G' = G + \{x, y\}$. G' consists of a cycle formed by the edge $\{x, y\}$ and the path from x to y in G .

(4 \rightarrow 5). Note that since $G + \{x, y\}$ creates a cycle for for any two non-adjacent vertices in G , it must be that there must be a path between x and y in G . We will now show that there is exactly one path between any two vertices in G . We will prove this by showing that if there are two vertices that have two different paths between them then G contains a cycle. Assume that there are two paths from u to v . Beginning at u , let a be the first vertex at which the two paths separate and let b be the first vertex after a where the two paths meet. Then, there are two simple paths from a to b with no common edges. Combining these two paths gives us a cycle.