

Mathematical Foundations of Computer Science

Lecture Outline

October 6, 2020

Graphs

A *graph* consists of two sets, a non-empty set, V , of vertices or nodes, and a possibly empty set, E , of 2-element subsets of V . Such a graph is denoted by $G = (V, E)$. Each element of E is called an *edge*. We say that an edge $\{u, v\} \in E$ *connects* vertices u and v . Two nodes u and v are *adjacent* if $\{u, v\} \in E$. Nodes adjacent to a vertex u are called *neighbors* of u . The number of neighbors of a vertex v is called the *degree* of v and is denoted by $\deg(v)$. The value $\delta(G) = \min_{v \in V} \{\deg(v)\}$ is the *minimum degree* of G , the value $\Delta(G) = \max_{v \in V} \{\deg(v)\}$ is the *maximum degree* of G . An edge that connects a node to itself is called a *loop* and multiple edges between the same pair of nodes are called *parallel edges*. Graphs without loops and parallel edges are called *simple graphs*, otherwise they are called *multigraphs*. Unless specified otherwise, we will only deal with simple graphs.

Example. Prove that the sum of degrees of all nodes in a graph is twice the number of edges.

Solution. Since each edge is incident to exactly two vertices, each edge contributes two to the sum of degrees of the vertices. The claim follows.

Example. In any graph there are an even number of vertices of odd degree.

Solution. Let V_e and V_o be the set of vertices with even degree and the set of vertices with odd degree respectively in a graph $G = (V, E)$. Then,

$$\sum_{v \in V} \deg(v) = \sum_{v \in V_e} \deg(v) + \sum_{v \in V_o} \deg(v)$$

The first term on R.H.S. is even since each vertex in V_e has an even degree. From the previous example, we know that L.H.S. of the above equation is even. Thus the second term on the R.H.S. must be even. Let $|V_o| = \ell$. We want to show that ℓ is even. Since each vertex in V_o has odd degree, we have

$$\begin{aligned} (2k_1 + 1) + (2k_2 + 1) + \cdots + (2k_\ell + 1) &\text{ is an even number} \\ 2(k_1 + k_2 + \cdots + k_\ell) + \ell &\text{ is an even number} \\ \therefore \ell &\text{ is an even number} \end{aligned}$$

This proves the claim.

A *walk* in G is a non-empty sequence $v_0e_0v_1e_1\dots e_{k-1}v_k$ of vertices and edges in G such that $e_i = \{v_i, v_{i+1}\}$ for all $i < k$. If the vertices in a walk are all distinct, we call it a *path* in G . Thus, a *path* in G is a sequence of distinct vertices $v_0, v_1, v_2, \dots, v_k$ such that for all i , $0 \leq i < k$, $\{v_i, v_{i+1}\} \in E$. The *length* of the walk (path) is k , the number of edges in the walk (resp. path). Note that the length of the walk (path) is one less than the number of vertices in the walk (path) sequence. If $v_0 = v_k$, the walk (path) is *closed*. A closed path is called a *cycle*.

The graph $H = (V', E')$ is a *subgraph* of $G = (V, E)$ if $V' \subseteq V$ and $E' \subseteq E$. A graph G is *connected* if there is a path in G between its every pair of vertices. A graph H is a *connected component* (“island”) of G if (a) H is a subgraph of G , (b) H is connected, and (c) H is maximal, i.e., H is not contained in any other connected subgraph of G . In short, H is a connected component of G if H is a maximal subgraph of G that is connected.

We say that H is an *induced subgraph* of a graph G if the vertex set of H is a subset of the vertex set of G , and if u and v are vertices in H , then (u, v) is an edge in H iff (u, v) is an edge in G .

Example. Prove that every graph with n vertices and m edges has at least $n - m$ connected components.

Solution. We will prove this claim by doing induction on m .

Induction Hypothesis: Assume that for some $k \geq 0$, every graph with n vertices and k edges has at least $n - k$ connected components.

Base Case: $m = 0$. A graph with n vertices and no edges has n connected components as each vertex itself is a connected component. Hence the claim is true for $m = 0$.

Induction Step: We want to prove that a graph, G , with n vertices and $k + 1$ edges has at least $n - (k + 1) = n - k - 1$ connected components. Consider a subgraph G' of G obtained by removing any arbitrary edge, say $\{u, v\}$, from G . The graph G' has n vertices and k edges. By induction hypothesis, G' has at least $n - k$ connected components. Now add $\{u, v\}$ to G' to obtain the graph G . We consider the following two cases.

Case I: u and v belong to the same connected component of G' . In this case, adding the edge $\{u, v\}$ to G' is not going to change any connected components of G' . Hence, in this case the number of connected components of G is the same as the number of connected components of G' which is at least $n - k > n - k - 1$.

Case II: u and v belong to different connected components of G' . In this case, the two connected components containing u and v become one connected component in G . All other connected components in G' remain unchanged. Thus, G has one less connected component than G' . Hence, G has at least $n - k - 1$ connected components.

Example. Prove that every connected graph with n vertices has at least $n - 1$ edges.

Solution. We will prove the contrapositive, i.e., a graph G with $m \leq n - 2$ edges is disconnected. From the result of the previous problem, we know that the number of components of G is at least

$$n - m \geq n - (n - 2) = 2$$

which means that G is disconnected. This proves the claim.

One could also have proved the above claim directly by observing that a connected graph has exactly one connected component. Hence, $1 \geq n - m$. Rearranging the terms gives us $m \geq n - 1$.