Mathematical Foundations of Computer Science
Lecture Outline
September 29, 2020

Example. Let n be a non-negative integer. Show that any $2^n \times 2^n$ region with one central square removed can be tiled using L-shaped pieces, where the pieces cover three squares at a time (Figure 1).

Solution. (Attempt 1) Let R_n denote a $2^n \times 2^n$ region. Let $P(n)$ be the property that R_n with one central square removed can be tiled using L-shaped pieces.

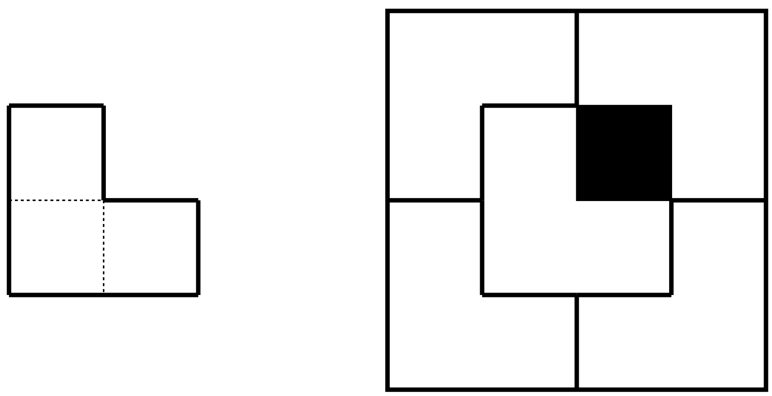

Figure 1: A L-tile and an L-tiling of a $2^2 \times 2^2$ region without a square.

Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 0$.
Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one central square removed requires 0 tiles.
Induction Step: We want to prove that $P(k+1)$ is true, i.e., region R_{k+1} with one central square removed can be tiled using L-shaped pieces.
R_{k+1} can be divided into four regions of size $2^k \times 2^k$. Note that the four central corners of R_{k+1} can be covered using one L-shaped tile and one square hole (Figure 2). Each of the four remaining regions has one hole and is of the size $2^k \times 2^k$. By induction hypothesis, these regions can be covered using L-shaped pieces. Thus, since the four disjoint regions can be covered using L-shaped tiles, R_{k+1} without a central square can also be covered using L-shaped tiles.
Our use of induction hypothesis is incorrect as we have assumed that region R_k without a *central* square (not a *corner* square) can be covered using L-shaped tiles.

Surprisingly, we can get around this obstacle by proving the following stronger claim.
"For all positive integers n, any R_n region with *any* one square removed can be L-tiled."

Let $P(n)$ be the property that R_n without one square can be L-tiled.
Induction Hypothesis: Assume that $P(k)$ is true for some k.

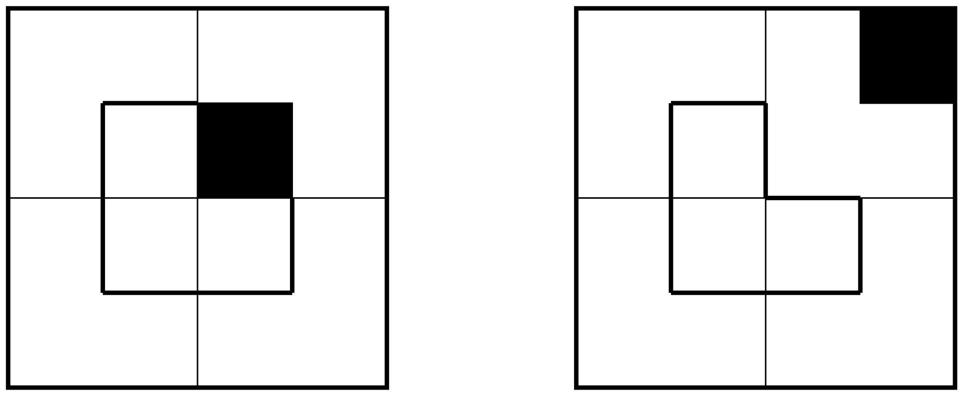

Figure 2: Illustration of the two proof attempts.

Base Case: We want to prove that $P(0)$ is true. This is true because a 1×1 region with one square removed requires 0 tiles.
Induction Step: We want to prove that $P(k+1)$ is true, i.e., region R_{k+1} wthout one square that is located anywhere can be L-tiled. Divide R_{k+1} into four R_k regions. One of the four R_k regions that does not have one square can be L-tiled (using induction hypothesis). Each of the other three R_k regions without the corner square that is located at the center of R_{k+1} can be L-tiled (using induction hypothesis). By using one more L-tile we can cover the three central squares of R_{k+1}.

Strong Induction.

For any property P, if $P(0)$ and $\forall n \in \mathbb{N}, P(0) \wedge P(1) \wedge P(2) \wedge \cdots \wedge P(k) \rightarrow P(k+1)$, then $\forall n \in \mathbb{N}, P(n)$.

Example. Prove that if n is an integer greater than 1 then either n is a prime or it can be written as a product of primes.

Solution. Let $P(n)$ be "n can be written as a product of primes".
Induction Hypothesis: Assume that $P(j)$ is true for $1 < j \leq k$.
Base Case: We want to show that $P(2)$ is true. This is clearly true as 2 is a prime.
Induction Step: We want to show that $P(k+1)$ is true.
Case I: $k+1$ is prime. In this case we are done.
Case II: $k+1$ is composite. Then,

$$k+1 = a \times b, \quad \text{for some } a \text{ and } b \text{ s.t. } 2 \leq a \leq b < k+1$$

By induction hypothesis, a is a prime or it can be written as a product of primes. The same applies to b. Since $k+1 = a \times b$, it can be written as a product of primes, namely those primes in the factorization of a and those in the factorization of b.

Example. Prove that, for any positive integer n, if $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parenthesis are inserted into their product, the number of multiplications used to compute the product is $n-1$.

Solution. Let $P(n)$ be the property that "If $x_1, x_2, \ldots, x_n$ are n distinct real numbers, then no matter how the parentheses are inserted into their product, the number of multiplications used to compute the product is $n-1$".
Induction Hypothesis: Assume that $P(j)$ is true for all j such that $1 \leq j \leq k$.
Base Case: $P(1)$ is true, since x_1 is computed using 0 multiplications.
Induction Step: We want to prove $P(k+1)$. Consider the product of $k+1$ distinct factors, $x_1, x_2, \ldots, x_{k+1}$. When parentheses are inserted in order to compute the product of factors, some multiplication must be the final one. Consider the two terms, of this final multiplication. Each one is a product of at most k factors. Suppose the first and the second term in the final multiplication contain f_k and s_k factors. Clearly, $1 \leq f_k, s_k \leq k$. Thus, by induction hypothesis, the number of multiplications to obtain the first term of the final multiplication is $f_k - 1$ and the number of multiplications to obtain the second term of the final multiplication is $s_k - 1$. It follows that the number of multiplications to compute the product of $x_1, x_2, \ldots, x_k, x_{k+1}$ is

$$(f_k - 1) + (s_k - 1) + 1 = f_k + s_k - 1 = k + 1 - 1 = k$$

Example. The game of NIM is played as follows: Some positive number of sticks are placed on the ground. Two players take turns, removing one, two or three sticks. The player to remove the last stick loses.

A winning strategy is a rule for how many sticks to remove when there are n left. Prove that the first player has a winning strategy iff the number of sticks, n, is not $4k+1$ for any $k \in \mathbb{N}$.

Solution. We will show that if $n = 4k+1$ then player 2 has a strategy that will force a win for him, otherwise, player 1 has a strategy that will force a win for him.

Let $P(n)$ be the property that if $n = 4k+1$ for some $k \in N$ then the first player loses, and if $n = 4k, 4k+2$, or $4k+3$, the first player wins. This exhausts all possible cases for n.
Induction Hypothesis: Assume that for some $z \geq 1$, $P(j)$ is true for all j such that $1 \leq j \leq z$.
Base Case: $P(1)$ is true. The first player has no choice but to remove one stick and lose.
Induction Step: We want to prove $P(z+1)$. We consider the following four cases.
Case I: $z+1 = 4k+1$, for some k. We have already handled the base case, so we can assume that $z+1 \geq 5$. Consider what the first player might do to win: he can remove 1, 2, or 3 sticks. If he removes one stick then the remaining number of sticks $n = 4k$. By strong induction, the player who plays at this point has a winning strategy. So the player who played first loses. Similarly, if the first player removes two sticks or three sticks, the remaining number of sticks is $4(k-1)+3$ and $4(k-1)+2$ respectively. Again, the first player loses (using induction hypothesis). Thus, in this case, the first player loses regardless of what move he/she makes.
Case II: $z+1 = 4k$, or $z+1 = 4k+2$, or $z+1 = 4k+3$. If the first player removes three

sticks in the first case, one stick in the second case, and two sticks in the third case then the second player sees $4(k-1)+1$ sticks in the first case and $4k+1$ sticks in the other two cases. By induction hypothesis, in each case the second player loses.

Example. Prove that the two forms of induction, weak induction and strong induction, are equivalent. In other words, prove that any statement that admits a strong induction proof can be proved using weak induction and vice-versa.

Solution. Suppose we want to show that a $P(n)$ is true for all positive integers $n \geq n_0$. The two forms of inductive proofs are as follows.

Weak Induction: Assume that

(a_w) $P(n_0)$ is true
(b_w) For any $k \geq n_0$, $P(k) \implies P(k+1)$ is true.

Then, $P(n)$ is true for all positive integers $n \geq n_0$.

Strong Induction: Assume that

(a_s) $P(n_0)$ is true
(b_s) For any $k \geq n_0$, $P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(k) \implies P(k+1)$ is true.

Then, $P(n)$ is true for all positive integers $n \geq n_0$.

We will show that it is always possible to convert a strong induction proof into a weak induction proof and vice-versa.

The conversion from a weak induction proof to a strong induction proof is trivial, since (b_s) implies (b_w).

We now show that a strong induction proof can be converted to a weak induction proof. Let

$$Q(n) \doteq P(n_0) \wedge P(n_0+1) \wedge \cdots \wedge P(n)$$

Induction Hypothesis: Assume that $Q(k)$ is true for some $k \geq n_0$.
Base Case: Since $Q(n_0) = P(n_0)$ and we know that $P(n_0)$ is true from (a_s), $Q(n_0)$ is true.
Induction Step: We want to show that $Q(k) \implies Q(k+1)$. We have

$$\begin{aligned} Q(k) &\implies P(k+1) \qquad \text{(from } (b_s)\text{)} \\ \therefore Q(k) &\implies Q(k) \wedge P(k+1) \\ \therefore Q(k) &\implies Q(k+1) \end{aligned}$$

Thus we have converted a strong induction proof in P to a weak induction proof in Q.