Mathematical Foundations of Computer Science
Lecture Outline
September 22, 2020

Example. Prove that the sum of the first n positive odd numbers is n^2.

Solution. We want to prove that $\forall$ positive integers $n, P(n)$ where $P(n)$ is the following property.

$$\sum_{i=0}^{n-1} 2i+1 = n^2$$

Base Case: We want to show that $P(1)$ is true. This is clearly true as

$$\sum_{i=0}^{0} 2i+1 = 1 = 1^2$$

Induction Hypothesis: Assume $P(k)$ is true for some integer $k \geq 1$.
Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to show that

$$\sum_{i=0}^{k} 2i+1 = (k+1)^2$$

We can do this as follows.

$$\begin{aligned}\sum_{i=0}^{k} 2i+1 &= \sum_{i=0}^{k-1} 2i+1+2k+1 \\ &= k^2+2k+1 \qquad \text{(using induction hypothesis)} \\ &= (k+1)^2\end{aligned}$$

Example. Show that for all integers $n \geq 0$, if $r \neq 1$,

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1}-1)}{r-1}$$

Solution. Let r be any real number that is not equal to 1. We want to prove that $\forall$ integers $n, P(n)$, where $P(n)$ is given by

$$\sum_{i=0}^{n} ar^i = \frac{a(r^{n+1}-1)}{r-1}$$

Base Case: We want to show that $P(0)$ is true.

$$\sum_{i=0}^{0} ar^i = a = \frac{a(r-1)}{r-1}$$

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k \geq 0$.
Induction Step: We want to show that $P(k+1)$ is true, i.e., we want to prove that

$$\sum_{i=0}^{k+1} ar^i = \frac{a(r^{k+2}-1)}{r-1}$$

We can do this as follows.

$$\begin{aligned}
\text{L.H.S.} &= \sum_{i=0}^{k+1} ar^i \\
&= \sum_{i=0}^{k} ar^i + ar^{k+1} \\
&= \frac{ar^{k+1}-a}{r-1} + ar^{k+1} \\
&= \frac{a(r^{k+1}-1)}{r-1} + \frac{ar^{k+1}(r-1)}{r-1} \\
&= \frac{a}{r-1}\left(r^{k+1}(1+r-1)-1\right) \\
&= \frac{a}{r-1}\left(r^{k+2}-1\right) \\
&= \frac{a(r^{k+2}-1)}{r-1}
\end{aligned}$$

Example. Prove that $\forall$ non-negative integers n,

$$\sum_{i=0}^{n} 2^i = 2^{n+1} - 1$$

Solution. By setting $a = 1$, $r = 2$ in the result of the previous problem, the claim follows.

Example. Prove that $\forall$ non-negative integers n, $2^{2n} - 1$ is a multiple of 3.

Solution. We want to prove that $\forall$ non-negative integers n, $P(n)$, where $P(n)$ is

$$2^{2n} - 1 = 3k, \text{ for some non-negative integer } k$$

Base Step: $P(0)$ is true as shown below.

$$2^0 - 1 = 0 = 3 \cdot 0.$$

Induction Hypothesis: Assume that $P(x)$ is true for some integer $x \geq 0$, i.e., $2^{2x} - 1 = 3 \cdot k'$, for some $k' \geq 0$.
Induction Step: We want to prove that $P(x+1)$ is true, i.e., we want to show that

$$2^{2(x+1)} - 1 = 3l, \text{ for some non-negative integer } l.$$

We can show this as follows.

$$\begin{aligned}
\text{L.H.S.} &= 2^{2(x+1)} - 1 \\
&= 2^{2x+2} - 1 \\
&= 2^{2x} \cdot 2^2 - 1 \\
&= 2^{2x} \cdot 4 - 1 \\
&= 2^{2x} \cdot (3+1) - 1 \\
&= 3 \cdot 2^{2x} + 2^{2x} - 1 \\
&= 3 \cdot 2^{2x} + 3 \cdot k' \qquad \text{(using induction hypothesis)} \\
&= 3(2^{2x} + k') \\
&= 3l, \qquad \text{where } l = 2^{2x} + k'
\end{aligned}$$

Since x and k' are integers l is also an integer. Hence, $P(x+1)$ is true.

Example. Prove that $\forall n \in \mathbb{N}, n > 1 \rightarrow n! < n^n$.

Solution. Below is a simple direct proof for this inequality.

$$\begin{aligned}
n! &= 1 \times 2 \times 3 \times \cdots \times n \\
&< n \times n \times n \times \cdots \times n \\
&= n^n
\end{aligned}$$

We now give a proof using induction. Let $P(n)$ denote the following property.

$$n! < n^n$$

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k > 1$.
Base Case: We want to prove $P(2)$. $P(2)$ is the proposition that $2! < 2^2$, or $2 < 4$, which is true.
Induction Step: We want to prove $P(k+1)$, i.e., we want to prove that $(k+1)! < (k+1)^{k+1}$.

$$\begin{aligned}
\text{L.H.S.} &= (k+1)! \\
&= k! \times (k+1) \\
&< k^k \times (k+1) \qquad \text{(using induction hypothesis)} \\
&< (k+1)^k \times (k+1) \quad \text{(since } k > 1\text{)} \\
&= (k+1)^{k+1}
\end{aligned}$$

Example. Recall that for any set A, $\mathcal{P}(A)$ denotes the power set of A. Let $S = \{x_1, x_2, \ldots, x_n\}$. Prove using induction that for all positive integers n, $|\mathcal{P}(S)| = 2^n$.

Solution. We will prove the claim using induction on n.
Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 1$. In other words, assume that if $S = \{x_1, x_2, \ldots, x_k\}$, then $|\mathcal{P}(S)| = 2^k$.
Base Case:$n = 1$. When $S = \{x_1\}$, there are exactly two subsets of S, namely $\emptyset$ and S, itself. Thus the claim is true when $n = 1$.
Induction Step: We want to prove that the claim is true when $n = k + 1$. In other words, we want to show that if $S = \{x_1, x_2, \ldots, x_k, x_{k+1}\}$, then $|\mathcal{P}(S)| = 2^{k+1}$. Let $S' = \{x_1, x_2, \ldots, x_k\}$. The set of all subsets of S can be partitioned into S_1 and S_2, where $S_1 \subset \mathcal{P}(S)$ contains subsets of S that does not contain x_{k+1}, and $S_2 \subset S$ contains subsets of $\mathcal{P}(S)$ that contains x_{k+1}. Thus we have

$$|\mathcal{P}(S)| = |S_1| + |S_2| \tag{1}$$

Note that S_1 contains all subsets of $\mathcal{P}(S')$. By the induction hypothesis, we have $|S_1| = |\mathcal{P}(S')| = 2^k$. We will now compute $|S_2|$. Observe that each set in S_2 is of the form $\{x_{k+1}\} \cup X$, where X is a subset of S'. By induction hypothesis, we know that there are 2^k subsets of S' and hence $|S_2| = 2^k$. Plugging in the values for $|S_1|$ and $|S_2|$ in (1), we get

$$|\mathcal{P}(S)| = 2^k + 2^k = 2^{k+1}$$

Example Let $A_1, A_2, \ldots, A_n$ be sets (where $n \geq 2$). Suppose for any two sets A_i and A_j either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. Prove by induction that one of these n sets is a subset of all of them.

Solution. We will prove the claim using induction on n.
Induction Hypothesis: Assume that the claim is true when $n = k$, for some integer $k \geq 2$. In other words, assume that if we have sets $A_1, A_2, \ldots, A_k$, where for any two sets A_i and A_j, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$ then one of the k sets is a subset of all of the k sets.
Base Case: $n = 2$. We have two sets A_1, A_2 and we know that $A_1 \subseteq A_2$ or $A_2 \subseteq A_1$. Without loss of generality assume that $A_1 \subseteq A_2$. Then A_1 is a subset of A_1 and is also a subset of A_2, so the claim holds when $n = 2$.
Induction Step: We want to prove the claim when $n = k + 1$. That is, we are given a set $S = \{A_1, A_2, \ldots, A_{k+1}\}$ of with the property that for every pair of sets $A_i \in S$ and $A_j \in S$, either $A_i \subseteq A_j$ or $A_j \subseteq A_i$. We want to show that there is a set in S that is a subset of all $k + 1$ sets in S. Let $S' = S \setminus \{A_{k+1}\}$. By induction hypothesis, there is a set $A_p \in S'$ that is a subset of all sets in S'. We now consider the following two cases.
Case 1: $A_p \subseteq A_{k+1}$. Then it follows that A_p is a subset of all sets in S.
Case 2: $A_{k+1} \subseteq A_p$. Since A_p is a subset of all sets in S' and $A_{k+1} \subseteq A_p$, it follows that A_{k+1} is a subset of all sets in S.

Example. For all $n \geq 1$, prove that n lines separate the plane into $(n^2+n+2)/2$ regions. Assume that no two of these lines are parallel and no three pass through a common point.

Solution. Let $P(n)$ be the property that n lines, such that no two of them are parallel and no three of them pass through a common point, separate the plane into $(n^2+n+2)/2$ regions. We will prove the claim by induction on n.

Induction Hypothesis: Assume that $P(k)$ is true for some integer $k > 0$.
Base Case: $P(1)$ is true since one line divides the plane into 2 regions which is also given by $(1^2+1+2)/2$.
Induction Step: To prove that $P(k+1)$ is true. Consider a set S of $k+1$ lines such that no two of them are parallel and no three of them pass through a common point. Remove any line ℓ from the set S. Let S' be the resulting set of k lines. By induction hypothesis, the k lines in S' divide the plane into $(k^2+k+2)/2$ regions. Now we add the line ℓ to the set S' to obtain the set S. Line ℓ intersects exactly once with each of the k lines in S'. These intersections divide the line ℓ into $k+1$ line segments. Each of these line segments passes through a region and hence $k+1$ additional regions are created. Hence, the total number of regions formed by $k+1$ lines is given by

$$\frac{k^2+k+2}{2} + k + 1 = \frac{k^2+3k+4}{2} = \frac{k^2+2k+1+k+3}{2} = \frac{(k+1)^2+(k+1)+2}{2}$$

Thus $P(k+1)$ is correct and this completes the proof.