

# Mathematical Foundations of Computer Science

## Lecture Outline

September 10, 2020

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### Permutations of Selected Elements.

We looked at permutations of  $n$  elements out of the available  $n$  elements. Now we will consider permutations of  $r$  elements out of the available  $n$  elements. Such an arrangement is called an  $r$ -permutation. For example,  $ab, ba, ac, ca, bc, cb$  are all 2-permutations of the set  $\{a, b, c\}$ .

Let  $P(n, r)$  denote the number of  $r$ -permutations of a set of  $n$  elements. What is the value of  $P(n, r)$ ?

Forming an  $r$ -permutation of a set of  $n$  elements can be thought of as an  $r$ -step process such that in step  $i, 1 \leq i \leq r$ , we choose the  $i$ th element of the ordering. There are  $n - (i - 1) = n - i + 1$  ways of performing step  $i$ . By the multiplication rule, the number of  $r$ -permutations equals

$$\begin{aligned} P(n, r) &= n \times n - 1 \times n - 2 \times \cdots \times n - (r - 1) \\ &= n \times n - 1 \times n - 2 \times \cdots \times n - r + 1 \\ &= \frac{n \times (n - 1) \times \cdots \times (n - r + 1) \times (n - r) \times \cdots \times 1}{n - r \times (n - r - 1) \times (n - r - 2) \times \cdots \times 1} \\ &= \frac{n!}{(n - r)!} \end{aligned}$$

**Example.** How many ways are there to select a first-prize winner, a second-prize winner, and a third-prize winner from 100 different contestants?

**Solution.** Selecting the winners can be done in 3 steps with each step  $i, 1 \leq i \leq 3$  choosing the winner in the  $i$ th place. Step  $i$  can be performed in  $100 - (i - 1)$  ways. By multiplication rule, the total number of possible ways in which the prizes can be given is  $100 \times 99 \times 98 = 970200$ . Note that this is same as  $P(100, 3)$ .

**Example.** In how many ways can we order 26 letters of the alphabet so that no two of the vowels  $a, e, i, o, u$  occur consecutively?

**Solution.** The task of ordering the letters so that no two vowels appear consecutively can be performed in two steps.

Step 1. Order the 21 consonants.

Step 2. Choose locations for the 5 vowels. The vowels can be placed before the consonants, between the consonants and after the consonants.

Step 1 can be performed in  $21!$  ways. To count the number of ways of performing Step 2, observe that there is only one location for placing a vowel before and after the consonants, and 20 locations for placing the vowels between the consonants. This gives a total of 22 valid locations for placing 5 vowels. Thus the number of ways of placing the 5 vowels in 5 of the 22 locations is  $P(22, 5)$ . This is because there are 22 locations for  $a$ , 21 for  $e$ , 20 for  $i$ , 19 for  $o$ , and 18 for  $u$ . By multiplication rule, the total number of orderings in which no two vowels occur consecutively equals

$$21! \times P(22, 5) = \frac{21! \times 22!}{17!}$$

## The Inclusion-Exclusion Formula.

If  $A, B$ , and  $C$  are any finite sets, then

$$\begin{aligned} |A \cup B| &= |A| + |B| - |A \cap B| \\ |A \cup B \cup C| &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| \\ &\quad + |A \cap B \cap C| \end{aligned}$$

If we have finite sets  $A_1, A_2, \dots, A_n$  then

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{i=1}^n |A_i| - \sum_{\substack{i,j \\ i < j}} |A_i \cap A_j| + \sum_{\substack{i,j,k \\ i < j < k}} |A_i \cap A_j \cap A_k| - \dots + (-1)^{n-1} |\cap_{i=1}^n A_i|$$

Observe that if the sets  $A, B$ , and  $C$  are mutually disjoint, i.e.,  $A \cap B = A \cap C = B \cap C = \emptyset$  then we get

$$\begin{aligned} |A \cup B| &= |A| + |B| \\ |A \cup B \cup C| &= |A| + |B| + |C| \end{aligned}$$

This is often called the *addition rule* or the *sum rule*.

**Example.** In how many ways can we select two books from different subjects among five distinct computer science books, three distinct math books, and two distinct art books?

**Solution.** The set of all possible two books from different subjects can be partitioned into three subsets,  $S_1, S_2$ , and  $S_3$ . The subset  $S_1$  contains two books belonging to computer science and math, the subset  $S_2$  contains two books belonging to computer science and art, and the subset  $S_3$  contains two books belonging to math and art. We have

$$\begin{aligned} |S_1| &= 5 \times 3 = 15 \\ |S_2| &= 5 \times 2 = 10 \\ |S_3| &= 3 \times 2 = 6 \end{aligned}$$

By the addition rule, total number of ways of selecting 2 books from different subjects equals  $|S_1| + |S_2| + |S_3| = 31$ .

**Example.** A PIN is typically made of four symbols chosen from 26 letters of the alphabet and the 10 digits, with repetitions allowed. How many PINS contain repeated symbols?

**Solution.** Let  $S$  denote the set of all possible PINs of four alpha-numeric characters. Let  $S_1$  denote the set of all possible PINs of four alpha-numeric characters with no repeated symbols. Let  $S_2$  denote the set of all possible PINs of four alpha-numeric characters with some symbols repeated. By the addition rule,

$$|S| = |S_1| + |S_2|$$

By simple application of multiplication rule, we see that  $|S| = 36^4 = 1679616$  and  $|S_1| = 36 \times 35 \times 34 \times 33 = 1413720$ . Plugging these values in the above equation, we get  $|S_2| = 265896$ .

**Example.** (a) How many integers from 1 through 1000 are multiples of 3 or multiples of 5?

(b) How many integers from 1 through 1000 are neither multiples of 3 nor multiples of 5?

**Solution.** (a) Let  $S = \{1, 2, 3, \dots, 1000\}$ . Let  $M \subseteq S$  be the set of integers that are multiples of 3 or multiples of 5. Let  $M_1 \subseteq S$  be the set of integers that are multiples of 3. Let  $M_2 \subseteq S$  be the set of integers that are multiples of 5. Note that the first integer in  $S$  that is divisible by 3 is  $3 = 3 \times 1$ . The last integer in  $S$  that is divisible by 3 is  $999 = 3 \times 333$ . Thus,  $|M_1| = 333$ . Similarly,  $|M_2| = 200$ . Note that  $M_1$  and  $M_2$  are not disjoint, i.e., there are integers like 15 that are divisible by 3 and by 5 and hence exist in  $M_1$  as well as  $M_2$ . We have double-counted them. So now, let's find the size of the set  $M_1 \cap M_2$ . Observe that each element in  $M_1 \cap M_2$  must be a multiple of  $3 \times 5 = 15$ . The first number in  $S$  that is a multiple of 15 is  $15 = 15 \times 1$  and the last number in  $S$  that is a multiple of 15 is  $990 = 15 \times 66$ . Thus,  $|M_1 \cap M_2| = 66$ . By the inclusion-exclusion formula, we get

$$|M| = |M_1| + |M_2| - |M_1 \cap M_2| = 333 + 200 - 66 = 467$$

(b) Let  $N \subseteq S$  be the set of integers that are neither multiples of 3 nor multiples of 5. Note that the sets  $M$  and  $N$  form a partition of the set  $S$ . Applying the addition rule we get

$$\begin{aligned} |S| &= |M| + |N| \\ \therefore |N| &= |S| - |M| \\ &= 1000 - 467 \\ &= 533 \end{aligned}$$

**Example.** How many strings are there of four lower-case letters that have the letter  $x$  in them?

**Solution.** Let  $S$  be the set of all possible four-letter strings that can be constructed using lower-case letters. The set  $S$  can be partitioned into two sets  $S_1$  and  $S_2$  where  $S_1$  is the set of all strings that contain at least one  $x$  and  $S_2$  is the set of strings that do not contain  $x$ . Hence we have

$$|S| = |S_1| + |S_2| \quad (1)$$

Each string in  $S$  and  $S_2$  can be constructed using the following four steps. In Step  $i$ ,  $1 \leq i \leq 4$ , we choose the letter in the  $i$ th location of the string.

While constructing a string in  $S$  each of the four steps can be performed in 26 ways. While constructing a string in  $S_2$  each of the four steps can be performed in 25 ways. Thus  $|S| = 26^4$  and  $|S_2| = 25^4$ . Substituting these values in equation (1) we get

$$|S_1| = 26^4 - 25^4 = 66351$$

**Incorrect Solution.** Here is an incorrect solution. Can you figure out what is wrong?

A four letter string that contains  $x$  can be constructed in two steps as follows. In Step 1 we choose one of the four positions for  $x$  (4 ways of doing this). In Step 2 we choose three letters for the remaining three places ( $26^3$  ways of doing this). By the multiplication rule, there are  $4 \cdot 26^3 = 70304$  four letter strings that contain  $x$ .

**Example.** How many even 4-digit numbers have no repeated digits?

**Solution.** Let  $S$  be the set of all 4-digit numbers with distinct digits. Let  $S_0$  be a set that contains all 4-digit numbers with distinct digits that end in a zero. Let  $S_1$  be the set of all 4-digit numbers with distinct digits that end in 2, 4, 6, 8. Note that the sets  $S_0$  and  $S_1$  partition the set  $S$  and hence we have

$$|S| = |S_0| + |S_1|$$

The procedure for constructing a number in  $S_0$  is as follows: in step 1, we choose the digit in position 4, in steps 2,3,4, we choose the digits in positions 1,2,3, respectively. There is only 1 way to do step 1, 9 ways to do step 2, 8 ways to do step 3, and 7 ways to do step 4. By the Multiplication Rule,  $|S_0| = 1 \times 9 \times 8 \times 7 = 504$ .

A number in  $S_1$  can be constructed similarly and hence  $|S_1| = 4 \times 8 \times 8 \times 7 = 1792$ .

Hence,  $|S| = 504 + 1792 = 2296$ .

## Combinations.

Let  $n$  and  $r$  be non-negative integers. An  $r$ -combination of a set of  $n$  elements means an unordered selection of  $r$  of the  $n$  elements of  $S$ . The symbol  $\binom{n}{r}$  (read as “ $n$  choose  $r$ ”) denotes the number of  $r$ -combinations of a set of  $n$  elements. This is same as the number of subsets of size  $r$  that can be chosen from a set of  $n$  elements.

The following numbers can be verified easily.

$$\binom{n}{r} = \begin{cases} 0 & \text{if } r > n \\ 1 & \text{if } r = 0 \text{ or } r = n \\ n & \text{if } r = 1 \end{cases}$$

Do you see the distinction between a  $r$ -permutation and a  $r$ -combination? A  $r$ -permutation is an *ordered* selection of  $r$  elements, i.e., both, which  $r$  elements, as well as the order in which they are chosen are important. Two  $r$ -permutations are the same if the  $r$  elements chosen are the same and they are chosen in the same order. In contrast, in a  $r$ -combination, only the choice of  $r$  elements is important. The order in which the  $r$  elements are chosen is irrelevant. Two  $r$ -combinations are the same if they have the same  $r$  elements regardless of the orders of selection of these elements.

In general, what is the value of  $\binom{n}{r}$ , i.e., how many  $r$ -combinations are possible if we have a set of  $n$  distinct objects?

We will answer this question by giving an expression that relates  $\binom{n}{r}$  and  $P(n, r)$ . A  $r$ -permutation can be obtained in two steps as follows.

Step 1. Choose  $r$  elements from the available  $n$  elements.

Step 2. Arrange the chosen  $r$  elements.

Step 1 can be performed in  $\binom{n}{r}$  ways. Step 2 can be performed in  $r!$  ways. By the multiplication rule, the total number of  $r$ -permutations is given by

$$P(n, r) = \binom{n}{r} \times r!$$

Rearranging the terms of the above equation we get

$$\binom{n}{r} = \frac{P(n, r)}{r!} = \frac{n!}{r!(n-r)!}$$

**Example.** We have a pool of 14 players from which 11 players must be chosen to play a cricket match? How many 11-member teams are possible?

**Solution.** The number of distinct 11-member teams is the same as the number of subsets of size 11 from the set of 14 players. This is given by

$$\binom{14}{11} = \frac{14!}{11!3!} = \frac{12 \times 13 \times 14}{1 \times 2 \times 3} = 364.$$

**Example.** Consider a set of twenty-five points, no three of which are collinear. How many straight lines do they determine? How many triangles do they determine?

**Solution.** Since no three points lie on a straight line, every two points determine a straight line. The number of straight lines equals the number of 2-combinations of a 25-element set. This is given by

$$\binom{25}{2} = \frac{25!}{2!23!} = \frac{24 \times 25}{1 \times 2} = 300.$$

Similarly every three points determine a triangle. Thus the number of triangles is given by

$$\binom{25}{3} = \frac{25!}{3!22!} = \frac{23 \times 24 \times 25}{1 \times 2 \times 3} = 2300.$$

**Example.** From a group of 8 women and 6 men, how many different committees consisting of 3 women and 2 men can be formed? What if 2 of the men are feuding and refuse to serve on the committee together?

**Solution.** The procedure of forming a committee of 3 women and 2 men is as follows.

Step 1. Choose the 3 women.

Step 2. Choose the 2 men

Step 1 can be done in  $\binom{8}{3}$  ways. Step 2 can be done in  $\binom{6}{2}$  ways. Using the multiplication rule, the total number of possible committees is  $\binom{8}{3} \times \binom{6}{2} = 840$ .

The second part of the question can be solved as follows. Let  $S_1$  be the set of all possible committees that do not contain the two feuding men. Let  $S_2$  be the set of all possible committees that contain exactly one of the two feuding men. Clearly, the no. of possible committees that do not contain the two feuding men together equals  $|S_1| + |S_2|$ . Using the reasoning used in the first part of the question we get  $|S_1| = \binom{8}{3} \times \binom{4}{2} = 336$  and  $|S_2| = 2 \binom{8}{3} \times \binom{4}{1} = 448$ . Hence the total number of committees without the two feuding men together is  $336 + 448 = 784$ .

The answer to the second part could also be derived by finding the number of all possible committees and then subtracting the number of committees in which the two feuding men are together. There are  $\binom{8}{3} \binom{6}{2} = 840$  committees in all out of which  $\binom{8}{3} \binom{2}{2} = 56$  committees contain the two feuding men. Thus there are  $840 - 56 = 784$  committees in all that have non-feuding men.

**Example.** There are 15 students enrolled in a course, but exactly 12 students attend on any given day. The classroom for the course has 25 distinct seats. How many different classroom seatings are possible?

**Solution.** A classroom seating can be constructed in two steps as follows.

Step 1. Choose 12 students out of 15 that are enrolled.

Step 2. Arrange 12 students in 25 distinct seats available.

Step 1 can be performed in  $\binom{15}{12}$  ways. Step 2 can be performed in  $P(25, 12)$  ways. By the multiplication rule, the number of different classroom seatings possible is given by

$$\binom{15}{12} \times P(25, 12) = \frac{15!}{12!3!} \times \frac{25!}{13!}$$

**Example.** How many 8-letter strings can be constructed by using the 26 letters of the alphabet if each string contains 3, 4, or 5 vowels? There is no restriction on the number of occurrences of a letter in the string.

**Solution.** Let  $E$  be the set of 8-letter strings that contain at least 3 vowels. Let  $E_i$  be the set of 8-letter strings containing exactly  $i$  vowels.

An element of  $E_i$ , i.e., a 8-letter string with exactly  $i$  vowels, can be constructed using the following steps.

Step 1. Choose  $i$  locations out of the available 8 locations for vowels.

Step 2. Choose the vowels for each of the  $i$  locations.

Step 3. Choose the consonants for each of the remaining  $8 - i$  locations.

Step 1 can be performed in  $\binom{8}{i}$  ways. Step 2 can be performed in  $5^i$  ways. Step 3 can be performed in  $21^{8-i}$  ways. By the multiplication rule, the number of 8-letter strings with exactly  $i$  vowels is given by

$$|E_i| = \binom{8}{i} 5^i 21^{8-i}$$

Since the sets  $E_3, E_4$ , and  $E_5$  partition the set  $E$ , by the addition rule we get

$$|E| = \sum_{i=3}^5 |E_i| = \sum_{i=3}^5 \binom{8}{i} 5^i 21^{8-i}$$

The following question was raised in class. What if we want to count all 8-letter strings with distinct letters that have 3, 4, or 5 vowels? In this case, the above procedure still applies. However, the number of ways of doing each step changes. Step 1 can be performed in  $\binom{8}{i}$  ways. Step 2 can be performed in  $P(5, i)$  ways. Step 3 can be performed in  $P(21, 8 - i)$  ways. By the multiplication rule, the number of 8-letter strings with distinct letters that have exactly  $i$  vowels is given by

$$\binom{8}{i} P(5, i) P(21, 8 - i)$$

The total number of 8-letter strings with distinct letters that have 3, 4, or 5 vowels is

$$\sum_{i=3}^5 \binom{8}{i} P(5, i) P(21, 8 - i)$$