Mathematical Foundations of Computer Science
Lecture Outline
November 19, 2020

Relations

A *binary relation* is a set of ordered pairs. For example, let $R = \{(1,2),(2,3),(5,4)\}$. Then since $(1,2) \in R$, we say that 1 is related to 2 by relation R. We denote this by $1\,R\,2$. Similarly, since $(4,7) \notin R$, 4 is not related to 7 by relation R, denoted by $4 \not R\ 7$.

A binary relation R from set A to set B is a subset of the cartesian product $A \times B$. When $A = B$, we say that R is a relation on set A.

Example. Let $A = \{1,2,3,4\}$ and $B = \{a,b,c\}$. Consider the following relations.

$$\begin{aligned}
R_1 &= \{(1,1),(1,2),(2,2),(2,3)\} \\
R_2 &= \{(1,2),(2,3),(3,4),(4,1),(4,4)\} \\
R_3 &= \{(1,a),(2,a),(3,b),(4,c)\} \\
R_4 &= \{(a,1),(a,3),(a,4),(c,1)\} \\
R_5 &= \{(a,a),(a,b),(1,c)\}
\end{aligned}$$

R_1 and R_2 are relations on A. R_3 is a relation from A to B. R_4 is a relation from B to A. R_5 is not a relation on sets A and B and it is neither a relation from A to B nor a relation from B to A.

Below are some more examples of relations.

- If S is a set then "is a subset of ", $\subseteq$ is a relation on $\mathcal{P}(S)$, the power set of S.
- "is a student in" is a relation from the set of students to the set of courses.
- "=" is a relation on $\mathbb{Z}$.
- "has a path in G to" is a relation on $V(G)$, the set of vertices in G.

Example. How many relations are there on a set A of n elements?

Solution. Note that any relation on A is a subset of $A \times A$ and since the power set of $A \times A$ contains all subsets of $A \times A$, the number of possible relations on A is the cardinality of the power set of $A \times A$. Since $|A \times A| = n^2$, the cardinality of the power set of $A \times A$ is 2^{n^2}. Thus our answer is 2^{n^2}.

Properties of Relations

Let R be a relation defined on set A. We say that R is

- *reflexive*, if for all $x \in A$, $(x, x) \in R$.
- *irreflexive*, if for all $x \in A$, $(x, x) \notin R$.
- *symmetric*, if for all $x, y \in A$, $(x, y) \in R \implies (y, x) \in R$.
- *antisymmetric*, if for all $x, y \in A$, $x\,R\,y$ and $y\,R\,x \implies x = y$.
- *transitive*, if for all $x, y, z \in A$, $x\,R\,y$ and $y\,R\,z \implies x\,R\,z$.

Note that the terms *symmetric* and *antisymmetric* are not opposites. A relation may be both symmetric and antisymmetric or can neither be symmetric nor be antisymmetric.

Example. What are the properties of the following relations?

R_1 : equality relation on $\mathbb{Z}$.
R_2 : "is a sibling of" relation on the set of all people.
R_3 : " $\leq$ " relation on $\mathbb{Z}$.
R_4 : " $<$ " relation on $\mathbb{Z}$.
R_5 : "$|$" relation on $\mathbb{Z}^+$.
R_6 : "$|$" relation on $\mathbb{Z}$.
R_7 : " $\subseteq$ " relation on the power set of a set S.
R_8 : $\{(x, y) \in \mathbb{R}^2 \,:\, |x - y| < \epsilon\}$, where $\epsilon = 0.001$

Solution.

$$\begin{aligned}
\text{Reflexive} &: R_1, R_3, R_5, R_7, R_8 \\
\text{Irreflexive} &: R_2, R_4 \\
\text{Symmetric} &: R_1, R_2, R_8 \\
\text{Antisymmetric} &: R_1, R_3, R_4, R_5, R_7 \\
\text{Transitive} &: R_1, R_3, R_4, R_5, R_6, R_7
\end{aligned}$$

Note that R_6 is not reflexive because $(0, 0) \notin R_6$; it is not antisymmetric because for any integer a, $a| - a$ and $-a|a$, but $a \neq -a$. R_2 is not transitive because x and z could be the same person. Observe that R_6 is an example of a relation that is neither symmetric nor antisymmetric. R_1 is an example of a relation that is symmetric and antisymmetric.

Example. How many reflexive relations are there on a set A of size n?

Solution. We know that $R \subseteq A \times A$. The procedure of constructing a reflexive relation R is as follows:

> Step 1: From $A \times A$, include in R all ordered pairs of the form (a, a).
> Step 2: For every ordered pair in $A \times A$ of the form (a, b), where $a \neq b$, choose whether to include it in R or not.

There is one way to do Step 1 and $2^{n(n-1)}$ ways to do Step 2. By the multiplication rule, the number of reflexive relations on a set n elements is $2^{n(n-1)}$.

Equivalence Relations

A relation R on a set A is an *equivalence relation* if and only if it is reflexive, symmetric and transitive.

Example Let m be a positive integer. Show that the *congruent modulo m* relation

$$R = \{(a, b) : a \equiv b \pmod{m}\}$$

is an equivalence relation on the set of integers.
(If m is a positive integer then integers x and y are *congruent modulo m*, written as $x \equiv y \pmod{m}$, if $m|(x - y)$).

Solution. To show that R is an equivalence relation we need to show that it is reflexive, symmetric, and transitive. R is reflexive because $a - a = 0$, and $0 = m \cdot 0$. R is symmetric because if $a \equiv b \pmod{m}$, it means that $a - b = m \cdot k$, for some integer k. Thus $b - a = m(-k)$ and hence $(b, a) \in R$. To show that R is transitive, suppose that that $a \equiv b \pmod{m}$ and $b \equiv c \pmod{m}$. Thus, for some integers q_1 and q_2, we have $a - b = m(q_1)$ and $b - c = m(q_2)$. Adding these two equations, we get $a - c = m(q_1 + q_2)$ and thus $a \equiv c \pmod{m}$. Hence R is transitive.

Example. Suppose that R is the relation on the set of strings of English letters such that $a\,R\,b$ if and only if $l(a) = l(b)$, where $l(x)$ is the length of the string x. Is R an equivalence relation?

Solution. R is reflexive as $l(a) = l(a)$, for any string a, and hence $a\,R\,a$. Next, suppose that $a\,R\,b$. This means that $l(a) = l(b)$ and hence $l(b) = l(a)$. Thus $b\,R\,a$ and hence R is symmetric. Finally, suppose that $a\,R\,b$ and $b\,R\,c$. Thus $l(a) = l(b)$ and $l(b) = l(c)$, which implies that $l(a) = l(c)$. Hence $a\,R\,c$ and R is transitive. Since R is reflexive, symmetric, and transitive, it is an equivalence relation.

Equivalence Classes

Let R be an equivalence relation on a set A and let $a \in A$. The *equivalence class of a*, denoted by $[a]_R$[1], is the set of all elements of A related (by R) to a; that is

$$[a]_R = \{x \in A \mid a\,R\,x\}$$

[1] The subscript R in $[a]_R$ is dropped when the relation in reference is clear from the context.

If $b \in [a]_R$, then b is called the *representative* of this equivalence class. Any element in a class can be used as a representative of the class.

Example. Let R be an equivalence relation on a set A. Then the following statements for elements $a, b \in A$ are equivalent

(i) $b \in [a]$ (ii) $[a] = [b]$ (iii) $[a] \cap [b] \neq \emptyset$

Solution. We will prove (i) $\Longrightarrow$ (ii), (ii) $\Longrightarrow$ (iii), and (iii) $\Longrightarrow$ (i).
(i) $\Longrightarrow$ (ii): We will prove the claim by showing that when $b \in [a]$, $[a] \subseteq [b]$ and $[b] \subseteq [a]$. Let c be any arbitrary but particular element in $[a]$. By definition, $a\,R\,c$. Since $b \in [a]$, it means that $a\,R\,b$, which further implies $b\,R\,a$ (since R is symmetric). Since R is transitive and we know that $b\,R\,a$ and $a\,R\,c$, we have $b\,R\,c$ and thus $c \in [b]$. We have thus proved that $[a] \subseteq [b]$.

Let $d \in [b]$. By definition, $b\,R\,d$. We also know that $a\,R\,b$. Since R is transitive, $a\,R\,b$ and $b\,R\,d$, we have $a\,R\,d$. Thus, by definition, $d \in [a]$. We have thus proved that $[b] \subseteq [a]$.

(ii) $\Longrightarrow$ (iii): To prove this we just need to show that $[a] \neq \emptyset$. Since R is reflexive, we know that $a \in [a]$. Since $[a] = [b]$ and $[a]$ is non-empty, it follows that $[a] \cap [b] \neq \emptyset$.

(iii) $\Longrightarrow$ (i): Let $c \in [a] \cap [b]$. Thus $a\,R\,c$ and $b\,R\,c$. Since R is symmetric, we have $c\,R\,b$. Since R is transitive, $a\,R\,c$ and $c\,R\,b$, we have $a\,R\,b$. By, definition $b \in [a]$.

Example. Let R be an equivalence relation on a set A. Then the set $\{[a]_R \mid a \in A\}$ is a partition of the set A. Each element of the set is called an *equivalence class* of R. Conversely, given a partition $\{A_i\}$ of the set A, there is an equivalence relation R that has sets A_i as its equivalence classes.

Solution. Since each element $a \in A$ is in its own equivalent class $[a]$, each equivalent class is non-empty and $\bigcup_{a \in A}[a] = A$. From the claim in the previous example (example we did in last class), for any two elements a and b in A, $[a]$ and $[b]$ are either equal or disjoint. Thus the equivalent classes partition the set A.

We now prove the converse. Let R be the relation on A that contains all possible pairs (x, y), where x and y belong to the same subset A_i in the partition. We want to show that R is reflexive, symmetric and transitive. R is reflexive as any element $a \in A$ is in the same subset of the partition as itself. Next suppose that $a\,R\,b$. This means that a and b are in the same subset of the partition of A. Thus, we have $b\,R\,a$ and hence R is symmetric. Finally, suppose that $a\,R\,b$ and $b\,R\,c$. This means that a and b are in the same subset of the partition and so are b and c. This means that a and c are in the same subset of the partition and hence we have $a\,R\,c$. Thus R is transitive.

Example. If an equivalence relation R is defined by the following set partition on A, then express R as a set of ordered pairs.

$$A = \{3, 4, 1\} \cup \{2\}$$

Solution.

$$R = \{(1,1),(2,2),(3,3),(4,4),(1,3),(1,4),(3,1),(3,4),(4,3),(4,1)\}$$