Mathematical Foundations of Computer Science
Lecture Outline
November 10, 2020

Graph Coloring

Consider the following scenario. There are n courses for which final exams need to be scheduled. Each exam needs a two hour slot. Since each student may be in more than one course, the exams need to be scheduled such that two courses that have common students don't have their final exams at the same time. The objective is to find minimum number of time slots that would be required to schedule all the exams.

A graph is *k-colorable* if each vertex can be colored using one of the k colors so that adjacent vertices are colored using different colors. The *chromatic number* of a graph G, $\chi(G)$, is the smallest value of k for which G is k-colorable.
The problem of scheduling exams can be modeled as a graph coloring problem. Construct a graph in which there is a vertex for each course and two vertices u and v are connected by an edge if there is a student who is taking both the courses corresponding to u and v. The chromatic number of the graph will provide the required solution to the problem.

Finding the chromatic number of a graph "quickly" is a very hard problem. Even finding a reasonable approximate solution is very hard!!

A *bipartite graph* is a graph that is 2-colorable.

Example. Prove that a graph with maximum degree at most k is $(k+1)$-colorable.

Solution. Let $P(n)$ be the property that a graph with n vertices and maximum degree at most k is $(k+1)$-colorable. We will now prove the claim by doing induction on n.
Base Case: $P(1)$ is clearly true as a graph with just one vertex has maximum degree zero and can be colored using one color.
Induction Hypothesis: Assume that $P(h)$ is true for some $h \geq 1$.
Induction Step: We want to prove that $P(h+1)$ is true. Let G be a graph with maximum degree at most k and having $h+1$ vertices. Let G' be the graph obtained from G by removing a vertex v along with the edges incident on v. G' has h vertices and has a maximum degree at most k. By induction hypothesis, G' is $(k+1)$-colorable. Now insert v along with its incident edges. Since we have a palette of $k+1$ colors and $deg(v) \leq k$, we can always color v using a color that is not used by any of its neighbors. Thus, $P(h+1)$ is true. This completes the proof.

Matchings

A *matching* in a graph is a set of edges with no shared end-points. The vertices incident on the edges of a matching M are called *M-saturated*, the others are called *M-unsaturated*. A *perfect matching* in a graph is a matching that saturates every vertex in the graph.

A *maximal matching* in a graph is a matching that is not contained in a larger matching. A *maximum matching* is a matching of maximum size among all matchings in the graph. Every maximum matching is a maximal matching, but the converse is not true. Figure 1 illustrates some of these definitions.

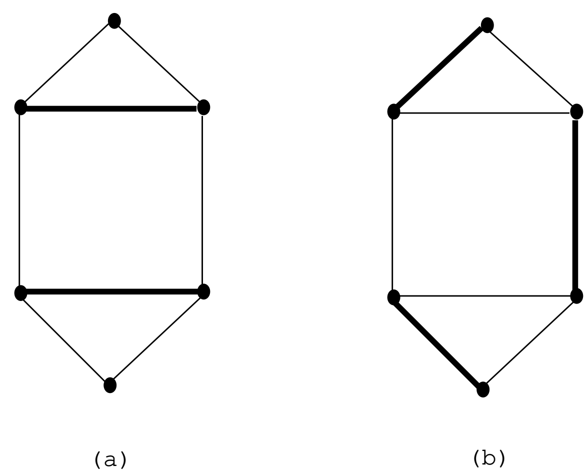

Figure 1: (a) a graph G with the bold edges representing a maximal matching, (b) the bold edges represent a maximum matching in G that is also perfect.

Given a matching, M, an *M-alternating path* is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are M-unsaturated is called an *M-augmenting* path. Given an M-augmenting path P, we can replace the edges of M in P with the edges in $E(P) \setminus M$ to obtain a new matching with one more edge. Thus, when M is a maximum matching there is no M-augmenting path.

For graphs G and H, the *symmetric difference* $G \oplus H$ is a subgraph of $G \cup H$ whose edges are the edges of $G \cup H$ that appear in either G or H, but not both. We also use the notation for set of edges; in particular, if M and M' are matchings then $M \oplus M' = (M \setminus M') \cup (M' \setminus M)$.

Example. Prove that a matching M in G is maximum iff G contains no M-augmenting path.

Solution. We will prove the necessary condition by proving its contrapositive, i.e., we will prove that if G contains an M-augmenting path then M is not a maximum matching. Suppose that G contains a M-augmenting path $v_0 v_1 v_2 \ldots v_{2m+1}$ (Note that an M-augmenting path must be of odd length). Define $M' \subseteq E$ by

$$M' = M \setminus \{(v_1, v_2), (v_3, v_4), \ldots, (v_{2m-1}, v_{2m})\} \cup \{(v_0, v_1), (v_2, v_3), \ldots, (v_{2m}, v_{2m+1})\}$$

Then M' is a matching in G and $|M'| = |M| + 1$. Thus M is not a maximum matching.

We will prove the converse by proving the contraposition. Assume that M is not a maximum matching. Let M' be a maximum matching in G. Then $|M'| > |M|$. Set $H = G[M \oplus M']$. Figure 2 illustrates this operation. Observe that every vertex in H has either degree one or degree two in H, since it can be incident with at most one edge of M and one edge of M'. Thus each component of H is either an even cycle with edges alternating in M and M' or else a path with edges alternating in M and M'. Since $|M'| > |M|$, H contains more edges of M' than of M, and H must contain a component which is a path, P, that starts and ends with edges in M'. Since the start vertex and end vertex of P are M'-saturated in H they must be M-unsaturated in G. Thus, P is an M-augmenting path in G. This completes the proof.

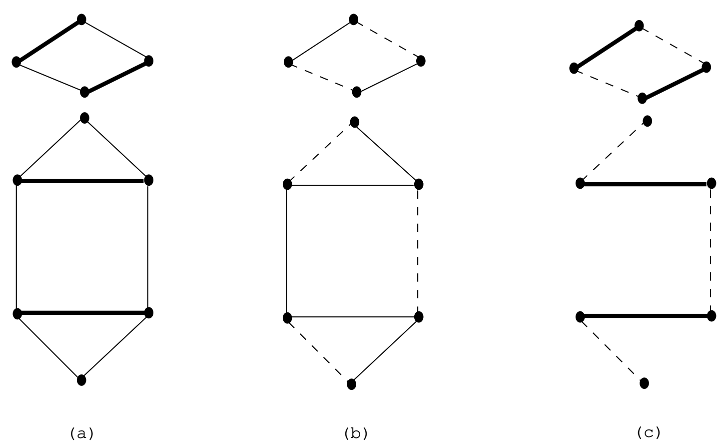

Figure 2: (a) a graph G with a matching M represented by the bold edges, (b) the dashed edges represent a matching M' in G, (c) $G[M \oplus M']$