Mathematical Foundations of Computer Science
Lecture Outline
November 3, 2020

Eulerian and Hamiltonian Graphs

An *Eulerian circuit* is a closed walk in which each edge appears exactly once. A connected graph is *Eulerian* if it contains an Eulerian circuit.

A *Hamiltonian cycle* in a graph G is a cycle in which each vertex of G appears exactly once. A graph is *Hamiltonian* if it contains a Hamiltonian cycle.

To determine whether a graph is Hamiltonian or not is significantly harder than determining whether a graph is Eulerian or not.

Example. If $\delta(G) \geq 2$ then G contains a cycle.

Solution. Let P be a longest path (actually, any *maximal* path suffices) in G and let u be an endpoint of P. Since P cannot be extended, every neighbor of u is a vertex in P. Since $deg(u) \geq 2$, u has a neighbor $v \in P$ via an edge that is not in P. The edge $\{u, v\}$ completes the cycle with the portion of P from v to u.

The following theorem gives us a necessary and sufficient condition for a connected graph to be Eulerian.

Example. Prove that a connected graph G is Eulerian iff every vertex in G has even degree.

Solution. *Necessity:* To prove that "if G is Eulerian then every vertex in G has even degree". Let C denote the Eulerian circuit in G. Each passage of C through a vertex uses two incident edges and the first edge is paired with the last at the first vertex. Hence every vertex has even degree.

Sufficiency: To prove that "if every vertex in G has even degree then G is Eulerian". We will prove this using induction on the number of edges, m.
Induction Hypothesis: Assume that the property holds for any graph G with j edges, for all j such that $0 \leq j \leq k$.
Base Case: $m = 0$. In this case G has only one vertex and that itself forms a Eulerian circuit.
Induction Step: We want to prove that the property holds when G has n vertices and $k+1$ edges. Since G has at least one edge and because G is connected and every vertex of G has even degree, $\delta(G) \geq 2$. From the result of the previous problem, G contains a cycle, say C. Let G' be the graph obtained from G by removing the edges in $E(C)$. Since C has either

0 or 2 edges at every vertex of G, each vertex in G' also has even degree. However, G' may not be connected. By induction hypothesis, each connected component of G' has an Eulerian circuit. We can now construct an Eulerian circuit of G as follows. Traverse C, but when a component of G' is entered for the first time, we detour along the Eulerian circuit of that component. The circuit ends at the vertex where we began the detour. When we complete the traversal of C, we have completed an Eulerian circuit of G.

Alternative Proof for the Sufficiency Condition: Let G be a graph with all degrees even and let

$$W = v_0 e_0 \ldots e_{l-1} v_l$$

be the longest walk in G using no edge more than once. Since W cannot be extended all edges incident on v_l are part of W. Since all vertices in G have even degree it must be that $v_l = v_0$. Thus W is a closed walk. If W is Eulerian then we are done. Otherwise, there must be an edge in $E[G] \setminus E[W]$ that is incident on some vertex in W. Let this edge be $e = \{u, v_i\}$. Then the walk

$$u e v_i e_i \ldots e_{l-1} v_l e_0 v_0 e_1 \ldots e_{i-1} v_i$$

is longer than W, a contradiction.

Example. For any integer $n \geq 3$, let G be a simple graph on n vertices, and assume that all vertices in G are of degree at least $n/2$. Prove that G has a Hamiltonian cycle.

Solution. Assume for contradiction that G does not have a Hamiltonian cycle. Add new edges to G one-by-one, until we come to a point where adding an edge, say (x, y), creates a Hamiltonian cycle. Let G' be the graph in which all vertices have degree at least $n/2$ and G' does not have a Hamiltonian cycle, but adding (x, y) will make G' Hamiltonian. Since adding edge (x, y) creates a Hamiltonian cycle in G', it must be that G' has a Hamiltonian path that begins at x and ends at y. Let the path be $x = v_1, v_2, \ldots, v_{n-1}, v_n = y$. We now apply the pigeon-hole principle as follows. Let the pigeons be the edges incident on the vertices x and y, and let the holes be the $(n-1)$ edges of the form (v_i, v_{i+1}), where $1 \leq i \leq n-1$. An edge (pigeon) of the form (x, v_i) is assigned to the "hole" (v_{i-1}, v_i) and an edge (pigeon) of the form (y, v_i) is assigned to the "hole" (v_i, v_{i+1}). Since $\deg(x) \geq n/2$ and $\deg(y) \geq n/2$ and at most one edge incident on x (or y) is assigned to a hole, by the pigeon-hole principle, there must be i such that $3 \leq i \leq n-1$ and there is an edge (x, v_i) and an edge (y, v_{i-1}) (see figure below). Note that since the edge (x, y) does not exist in G', the hole corresponding to (v_1, v_2) only has one edge, namely (x, v_2). Similarly, the hole (v_{n-1}, v_n) will only contain the edge (y, v_{n-1}). But this would mean that $x v_2 v_3 \cdots v_{i-1} y v_{n-1} v_{n-2} \cdots v_i$ is a Hamiltonian cycle, a contradiction.