This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit, all your answers should be carefully justified; in particular, please make sure to explicitly define your sample space for any probability question unless otherwise specified. Please see Piazza for the updated collaboration policy

Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

1. **[12 pts] I'm like TT (Three-Tree) just like TT**

    A *three-tree* is a tree such that all vertices have degree either 1 or 3.

    (a) If a three-tree T has ℓ leaves, show that it has $\ell - 2$ vertices of degree 3.

    (b) Let T be a three-tree with at least four leaves, that is, $\ell \geq 4$. Show that there is some internal vertex that is adjacent to two leaves.

2. **[12 pts] Kneed for Speed**

    Kara is going hiking with Matthew, Ziya, and Kadin because they are all very adventurous and realize that if they don't hike now, while they're young and able, then when? "You only have so many years with good knees," Kara says, echoing the wise words of her dear grandmother. Matthew, with suspiciously brittle and raw spaghetti-like knees, agrees.

    The group is trying to decide which of the n mountains in Arizona they should hike this weekend. The mountains are labeled 1 to n. Let S be the mountain they end up hiking. To convince herself that her knees are still full of youth and spontaneity, Kara decides on a whacky scheme to choose S. She'll visit each mountain in numerical order, i.e. starting at mountain 1, then 2, then 3 etc. until she visits mountain n. When she visits a mountain i, she has an independent $1/i$ chance of setting S to be mountain i (she may change the value of S multiple times). After Kara visits all the mountains, for any given i, what is $\Pr[S = i]$?

3. **[8 pts] AAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAAA**

    Abhishek, Adi, Andy, Anni, Anusha, Arnold, Atharva, and Ayotomiwa want to go on the longest hike ever - from Antarctica to Albuquerque, New Mexico. Right before they leave, Anusha realizes that the hike will proabably take a couple hours (at least), so she decides to pack some food for the whole group. More specifically, she packs 3 separate backpacks of healthy hiking snacks: the first backpack has 1 apple and 4 avocados, the second has 4 apples and 2 avocados, and the third has 2 apples and 5 avocados.

About halfway through the hike (in the Atacama Desert in Chile), Adi gets extremely hungry and uniformly at random picks 1 food to eat from each of the 3 backpacks, independently. What is the probability that he ate an apple from the first backpack, given that exactly 2 of the foods he ate were apples?

4. **[14 pts] Old TAs Hydrate, New TAs Diedrate**

   William Qian, William Li, and William Vauclain decide to hold a hiking challenge to determine who the alpha-William is. The losers will be banned from the 160 staff forever. The hiking challenge will be hosted by Sid. In order to help them through their hike, Sid has set up n water bottles for the Williams to choose from and take on their hike; however, since Sid is very lazy, he only filled up c of the water bottles with water. Sid is a lazy senior and likes to play the seniority card, so Sid decides to give an advantage to the older TAs. Since William Qian has been on staff the longest, Sid decides that William Qian will get to randomly choose exactly p of the n water bottles, where $p < c \leq n$. Since William Vauclain has been on staff the second longest, Sid decides that William Vauclain will randomly choose exactly 1 water bottle from the remaining $n - p$ water bottles. Since William Li is a brand new TA, Sid decides that William Li cannot select any water bottles (Sid: "unfortunately, hydration is only for old TAs!"). PS, this is why we need an alpha-William, this first name and last name thing is very difficult!

   (a) What is the probability that William Vauclain takes a water bottle with water in it?

   (b) Given that the William Vauclain took a water bottle with water in it, what is the probability that William Qian took at least one water bottle with water in it?

5. **[12 pts] Buy Low, Sell Hi(ke)**

   Quinn and Linda operate competing infrastructure services. The Governor asks them to create a hiking trail network $G = (V, \mathcal{E})$ that connects every mountain $v \in V$ in Pennsylvania to each other via a series of hiking trails, where each hiking trail $e \in \mathcal{E}$ is between two mountains and there is at most one trail between each pair of mountains. Since Quinn and Linda are cost-cutting capitalists, they want to minimize the number of trails needed so they each create a distinct and minimally-connected network of hiking trails. In other words, each of their plans is a spanning tree of G.

   Linda's plan consists of the spanning tree $T = (V, E)$, and Quinn's plan consists of the spanning tree $T' = (V, E')$. The Governor notices that for any hiking trail e such that $e \in E \setminus E'$, there is a trail $e' \in E' \setminus E$ such that $(T - e) + e'$ and $(T' + e) - e'$ are also valid spanning tree plans. Prove that the Governor is correct.

6. **[12 pts] Beans, beans, the magical fruit, the more you eat, etc...**

   Serena is getting ready for her hiking trip this weekend. However, in order to prepare, she needs

to have enough food so she doesn't starve! So, being ever ready, Serena needs to pick up n cans of beans from n separate, distinct stores, where n is a positive integer. Serena also notices that each pair of stores has exactly one one-way road between them. Prove that there is a way for Serena to travel along the roads such that she visits each of the n stores exactly once and never passes it again.