

Homework 7H

Due: 9am EDT October 22, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in []. To receive full credit, all your answers should be carefully justified; in particular, please make sure to explicitly define your sample space for any probability question unless otherwise specified. Please see Piazza for the updated collaboration policy. Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

1. [12 pts] One Bean Two Bean Red Bean Blue Bean

Anusha is eating from a bowl of red and blue jelly beans during office hours. At the end of her shift, there are strictly less than 9 jelly beans left in her bowl. Being quite full, Anusha randomly selects two more jelly beans to eat (each pair of jelly beans is equally likely to be picked). Both of the jelly beans happen to be red.

Anusha looks down and counts exactly how many jelly beans were in the bowl. Then, being a 160 TA, she does some quick math in her head and realizes the probability that both of the jelly beans she picked being red is exactly $\frac{1}{2}$. With this information, can you figure out how many of the $n < 9$ remaining jelly beans were blue and how many were red?

2. [15 pts] TAs Talking about Tomiwa's Tummy

Tomiwa has had one too many beans for dinner and is having bad acid reflux. Embarrassed, he asks Oliver for help. Instead of helping Tomiwa, Oliver told all of his friends on staff. Those TAs went on to tell all of their friends, who in turn tell all their friends, and so on. Suppose that there are $n \geq 2$ TAs other than Tomiwa, and that each TA is friends with at least $\frac{n}{2}$ other TAs. Prove that all n TAs eventually hear about Tomiwa's acid reflux.

3. [15 pts] Beef With a Side of Beans

Taki and Yuyang have had beef for as long as they can remember, but today they have decided to finally determine who is the better TA. To do this, Taki proposes that they have a 9-round bean-eating contest throughout a full 7-day week (for some reason, Taki thinks that this is the best way to determine who is a better TA). Note that all 9 rounds will be completed regardless of who won the previous rounds.

Since they are both so busy with schoolwork, Taki gives Yuyang a list of 42 timeslots throughout the week - 6 on each day - and asks Yuyang to choose 9 timeslots so that they can do one round of the bean-eating contest per timeslot. Knowing that Yuyang is equally likely to choose each of the timeslots (and he doesn't pick any twice), what is the probability that Taki and Yuyang

will do at least one round of the bean-eating contest on every day of the 7-day week?

4. [10 pts] When's the Last Time Richard Ate? It's Bean a Minute

Every night at around 3AM, Richard has a sudden urge to eat some beans. Luckily, he owns an underground bean power system with n bean stations and m tunnels that connect them, where n and m are both integers and $n \geq 1$. Each tunnel connects exactly two bean stations, and there's at most one tunnel between two bean stations. Prove that $n \leq n^2 - 2m$.

5. [18 pts] UBS (Uh-Bean-Shake)'s Revenge

Uh-Bean-Shake went to class one day with a big tub of beans. Saurabh wants to eat beans desperately, but Uh-Bean-Shake is currently angry with Saurabh for stealing the beans he brought to class the day before. The other TAs tell Saurabh that if he can solve the following proof, Uh-Bean-Shake will forgive him. Help Saurabh prove that for any natural number n , it is possible to select 2^n numbers from any arbitrary collection of 2^{n+1} natural numbers such that the sum of the 2^n numbers is divisible by 2^n . Note that not all the numbers are necessarily distinct.