

Homework 5T

Due: 9:00 am EDT, October 1, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in []. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed.

1. [15 pts] Weilin is an engineering genius, and has designed very fancy basketballs for the Finals. In order to inflate any set of n of these basketballs, the first basketball needs exactly 1 gram of air, the second needs exactly $1/2$ grams, the third $1/3$ grams, and so on, so that the k^{th} basketball needs exactly $1/k$ grams of air.

Weilin is also a math genius, and postulates that in order to inflate a set of 2^n basketballs, where n is a natural number, she will need at least $1 + \frac{n}{2}$ grams of air. That is, defining

$$H_n = \sum_{k=1}^n \frac{1}{k} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

Weilin postulates that $H_{2^n} \geq 1 + \frac{n}{2}$ for all natural numbers n . Prove Weilin's claim.

2. [15 pts] Atharva is devastated over his beloved Boston Celtics' loss to the Miami Heat in the Eastern Conference Finals. However, he still wants to watch the superior Heat take on the Los Angeles Lakers in Game 1 of the Finals. He was too busy burning his Jayson Tatum jersey to finish his homework, and he needs to finish it if he wants to watch the game. Can you help him determine whether each of these proofs is valid or invalid?

For each of the "proofs" below, say whether the proof is valid or invalid. If it is invalid indicate clearly as to where the logical error in the proof lies and justify why this is a logical error. If the proof is valid, you can simply say so. Just stating that the claim is false will not be awarded credit.

- (a) **Claim:** $\forall n \in \mathbb{Z}^+, n^2 \leq n$.

Proof:

Base Case: For $n = 1$, the claim is true since $1^2 = 1$.

Induction Hypothesis: Assume that $k^2 \leq k$, for some $k \in \mathbb{Z}^+$.

Induction Step: We need to show that

$$(k+1)^2 \leq k+1$$

We can see that

$$k^2 \leq k^2 + 2k = (k^2 + 2k + 1) - 1 = (k + 1)^2 - 1$$

Since $(k + 1)^2 \leq k + 1$,

$$(k + 1)^2 - 1 \leq (k + 1) - 1 = k$$

Thus we get $k^2 \leq k$, which we know is true by the induction hypothesis.

(b) **Claim:** $\forall n \in \mathbb{Z}^+, 5^n = 5$.

Proof: We will prove the claim using strong induction on n .

Base Case: For $n = 1$, the claim is true since $5^1 = 5$.

Induction Hypothesis: Assume that $5^j = 5$, for all integers j s.t. $1 \leq j \leq k$ for some $k \in \mathbb{Z}^+$.

Induction Step: We need to prove that $5^{k+1} = 5$, and we have

$$\begin{aligned} 5^{k+1} &= \frac{5^k 5^k}{5^{k-1}} \\ &= \frac{5 \cdot 5}{5} \quad (\text{using induction hypothesis}) \\ &= 5 \end{aligned}$$

This completes the strong induction proof, so $5^n = 5$, for all $n \in \mathbb{Z}^+$.

(c) **Claim:** For all negative integers n ,

$$(-2) + (-4) + \dots + (2n) = -n^2 + n$$

Proof: We will prove the claim using induction on n .

Base Case: The claim holds when $n = -1$ since $-2 = -(-1)^2 + (-1) = -1 - 1 = -2$.

Induction Hypothesis: Assume that $(-2) + (-4) + \dots + (2k) = -k^2 + k$, for some $k \in \mathbb{Z}$, $k \leq -1$.

Induction Step: We want to prove that the claim is true when $n = k - 1$. That is, we want to prove that

$$(-2) + (-4) + \dots + (2k) + (2(k - 1)) = -(k - 1)^2 + (k - 1)$$

$$\begin{aligned} \text{L.H.S.} &= (-2) + (-4) + \dots + (2k) + (2(k - 1)) \\ &= (-2) + (-4) + \dots + (2k) + (2k - 2) \\ &= -k^2 + k + (2k - 2) \quad (\text{using induction hypothesis}) \\ &= -k^2 + 3k - 2 \\ &= (-k^2 + 2k - 1) + (k - 1) \\ &= -(k - 1)^2 + (k - 1) \end{aligned}$$

This completes the induction proof.

(d) **Claim:** $\forall n \in \mathbb{N}, 5n = 0$.

Proof: We will prove the claim using strong induction on n .

Base Case: The claim holds when $n = 0$ since $5 \cdot 0 = 0$.

Induction Hypothesis: Assume that $5j = 0$, for all $0 \leq j \leq k$, for some $k \in \mathbb{N}$, $j \in \mathbb{Z}$.

Induction Step: We must show that $5(k + 1) = 0$. Let $k + 1 = a + b$, where $a > 0$ and $0 < b \leq k$ are integers. From the induction hypothesis we know that $5a = 0$ and $5b = 0$, therefore

$$5(k + 1) = 5(a + b) = 5a + 5b = 0 + 0 = 0$$

This completes the induction proof.

(e) **Claim:** $\forall n \in \mathbb{Z}^+$, if p and q are positive integers such that $\max(p, q) = n$, then $p = q$.

Proof:

Base Case: Let $n = 1$. If p and q are positive integers such that $\max(p, q) = 1$, then p and q must both be 1, satisfying the claim.

Induction Hypothesis: Assume that the statement holds for $n = k$, where k is an arbitrary positive integer. In other words, we assume that if p and q are positive integers such that $\max(p, q) = k$, then $p = q$, for some positive integer k .

Induction Step: Consider the case where $n = k + 1$, and let p' and q' denote two positive integers such that $\max(p', q') = k + 1$. Then we must have $\max(p' - 1, q' - 1) = k$, which, by our induction hypothesis, implies that $p' - 1 = q' - 1$, and hence that $p' = q'$. This proves our claim.