CIS 160 Assigned: September 24, 2020

Homework 4H

Due: 9 am EDT, September 29, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed. All your answers should be in closed form unless otherwise specified in the question. Please refer to pinned Piazza Post @341 for the definition of closed forms.

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1. [**10 pts**] Tien is looking for 3 rainbows out of the 40 around him to shine on his pots of gold. However, this fictitious reality, rainbows are made out of up to only 4 colors - red, orange, blue, and purple. 25 of them have red, 30 of them have orange, 33 of them have blue, and 35 of them have purple. Because Tien is picky about his rainbows, he wants the 3 shining in his pots of gold to have all four colors. Prove that there exists at least 3 rainbows such that each of the rainbows have all 4 of the possible colors.

2. [**24 pts**] Wow! A magnificent double rainbow has appeared outside! Unfortunately, Colin suffers from iridophobia (the fear of rainbows) and must bunker down in his windowless basement in order to escape the multi-colored monstrosity outside. To pass the time, he decides to do some proofs. Help him out by proving the following using induction.

(a) $\forall n \in \mathbb{N}$, $3^{3n+4} + 2^{n+2}$ is divisible by 5.

(b) Let $r, n \in \mathbb{N}$ and let $r \leq n$. Then

$$\binom{n+1}{r+1} = \sum_{k=r}^{n} \binom{k}{r}$$

(c) Let $n \in \mathbb{Z}^+$, and let p be a prime number. Prove by induction that n^p can be expressed as the sum of n and some multiple of p.

3. [**10 pts**] Give a combinatorial proof to show that for all integers $n \geq 3$,

$$3^{n-2} \cdot n \cdot (n-1) = \sum_{k=2}^{n} \binom{n}{k} (k)(k-1)2^{k-2}$$

4. [**8 pts**] Yuyang is notorious for being the staff sad boi. To help Yuyang feel less blue and lonely, Daniel decides to gift him a comfort package containing exactly r Care Bears. Daniel goes to the CareBears website and sees that there are k colors of Care Bears, with an infinite number of each color besides the yellow bear (aka the "Funshine Bear"), of which there is only 1. How many ways can Daniel create a comfort package from Care Bears to keep Yuyang company? Note that bears of the same colors are indistinguishable.

5. [8 pts] Over quarantine, Big Brain Billy developed an obsession with Skittles and collected 160 packs of Skittles. Tiffany, upon seeing Billy's hoard, now also wants to "taste the rainbow" and asked Billy to share his Skittles. Fortunately for her, Billy is feeling generous today and will give her 1 pack of Skittles if she can prove the below claim:

> We define A to be a n-subset of B if $A \subseteq B$ and $|A| = n$. Let S be the set of all n-subsets of the set $\{1, 2, 3, \ldots, 2n\}$. Assume that $n \geq 2$. Prove that $|S|$ is composite.

Help Tiffany "taste the rainbow" by proving the claim!

6. [10 pts] Lucy wants to bring together all of her friends for a *Mario Kart* Tournament! She wants to make sure that out of all of the people that compete in this tournament, there will be at least one "karting champion". There will be $n \geq 1$ participants, and each participant in the tournament will go up against every other participant exactly once on the most difficult course–Rainbow Road–in a one-on-one race with one clear winner and one clear loser (no ties!).

Note that victories are not transitive – participant a beating participant b and participant b beating participant c does not imply that a will have beat c, as well.

Lucy sets the following criteria for a participant to be a "karting champion": a participant x is a "karting champion" if for all other participants y, either x beat y or x beat some third participant z who beat y.

Help Lucy plan her tournament and prove that at least one of the n participants will be a "karting champion".