CIS 160 Assigned: September 15, 2020

Homework 3T

Due: 9:00 am EDT, September 17, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed. All your answers should be in closed form unless otherwise specified in the question. Please refer to pinned Piazza Post @341 for the definition of closed forms.

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1. [12 pts] Answer the following questions. No justification is necessary; however, if your answer is incorrect, you may receive some partial credit for nearly correct justification.

(a) Oliver is the star quarterback of the CIS 160 football team. He throws 63 touchdown passes. Each of the touchdown passes is caught by one of his two star wide receivers, Olivia C or Olivia Z. At least one of his touchdown passes is caught by Olivia Z. How many ways can this have occurred?

(b) Unfortunately, Will L and Will V were not good enough to make the CIS 160 football team and are now the team managers of the team. There are 99 players on the team each with distinct jersey numbers from the set $\{1, 2, \ldots, 99\}$. Will L and Will V need to order the players in a line such that the sum of the jersey numbers of two consecutive players in the line is odd. How many ways can they do this?

(c) Linda L and Linda W are the team captains of the CIS 160 football team. In order to prepare for the season, Linda L and Linda W want to arrange a special scrimmage which requires 24 TAs. The Lindas split the 24 TAs into 12 TAs on the offense and 12 TAs on defense. Then within both the offense and defense, the Lindas assign two TAs to 6 different positions. How many ways can they organize the 24 TAs into offense, defense, and positions?

2. [10 pts] Your great-grandparents, Richard and Jared, are fighting over the remote. Richard really wants to watch a football game from 1967, but Jared wants to watch a game from 1987. However, they come to an agreement: if you can prove that for any positive integer a such that $a = 19c^2$, where $c \in \mathbb{N}$, that the $\sqrt{a}$ is irrational, then Richard and Jared will watch the game from 1987. Please provide a proof to this claim so your great-grandparents stop fighting. (You may assume that the square root of 19 is irrational without proof).

3. [8 pts] For each of the "proofs" below, say whether the proof is valid or invalid. If it is invalid, indicate clearly as to where the logical error in the proof lies, and justify why this is a logical error. If the proof is valid, you may simply say so. Just stating that the claim is false will not be awarded credit.

(a) **Claim:** All natural numbers are divisible by 143.

Proof: Suppose, for the sake of contradiction, the statement were false. Let X be the set of counterexamples, i.e., $X = \{x \in \mathbb{N} \mid x \text{ is not divisible by } 143\}$. The supposition that the statement is false means that $X \neq \emptyset$. Since X is a nonempty set of natural numbers, it contains a least element z.

Note that $0 \notin X$ because 0 is divisible by 143. So $z \neq 0$. Now consider $z - 143$. Since $z - 143 < z$ (and z is the smallest counterexample) then $z - 143$ is not a counterexample to the original statement and is therefore not in X. Therefore $z - 143$ is divisible by 143; that is, there is an integer a such that $z - 143 = 143a$. So $z = 143a + 143 = 143(a + 1)$ and z is divisible by 143, contradicting $z \in X$.

(b) **Claim:** For all $n \in \mathbb{N}, 2n + 1$ is a multiple of 3 $\implies$ $(n^2 + 1)$ is a multiple of 3.

Proof: We will prove the contrapositive. Assume $(2n + 1)$ is not a multiple of 3.

- If $n = 3k$, for $k \in \mathbb{N}$, then $n^2 + 1 = 9k^2 + 1$ is not a multiple of 3.
- If $n = 3k + 1$ for $k \in \mathbb{N}$, then $(2n + 1) = 6k + 3$ is a multiple of 3, so the original claim holds, as false implies everything.
- If $n = 3k + 2$ for $k \in \mathbb{N}$, then $n^2 + 1 = 9k^2 + 12k + 5$ is not a multiple of 3.

In all cases, we have concluded $n^2 + 1$ is not a multiple of 3, so we have proved the claim.

(c) **Claim:** $\sqrt{15} + \sqrt{31} < 12$

Proof: Squaring both sides of the inequality in question gives us $46 + 2\sqrt{465} < 144$ which further simplifies into $\sqrt{465} < 49$. Squaring both sides gives us $465 < 2401$, which is true. Therefore, $\sqrt{15} + \sqrt{31} < 12$ must be true.

(d) **Claim:** If x and y are integers then $4xy^3$ has a different parity than x.

Proof: Assume, without loss of generality, that x is odd. By definition of an odd integer, $x = 2k + 1$, for some integer k. Thus:

$$4xy^3 = (2k + 1)(4y^3) = 8ky^3 + 4y^3 = 2(4ky^3 + 2y^3)$$

Since $4ky^3 + 2y^3$ is an integer, $4xy^3$ is even and hence has a different parity than x.