CIS 160 Assigned: September 17, 2020

Homework 3H

Due: 9:00 am EDT, September 22, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit all your answers should be carefully justified. Each solution must be written independently by yourself - no collaboration is allowed. All your answers should be in closed form unless otherwise specified in the question. Please refer to pinned Piazza Post @341 for the definition of closed forms.

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1. [12 pts] The NFL season has finally begun, and to celebrate, Jasmine, Jason, and Jonathan are creating a fantasy football league for the TAs! In the fantasy league, there will be $n > 0$ teams. After the TAs all draft their picks, each team i has x_i players (note, there can't be fewer than zero players in a team, and teams don't need to have the same number of players). We define $\overline{x} = (x_1 + x_2 + \ldots + x_n)/n$ to be the average number of players per fantasy team, and we define $P = \{x_1, x_2, \ldots, x_n\}$.

(a) Prove that $\exists\ i$ s.t. $x_i \geq \overline{x}$.

(b) Let $T = \{x \mid x \in P, x > 2\overline{x}\}$. Prove that $|T| < \dfrac{|P|}{2}$.

2. [10 pts] Quinn recently learned that the first 1-billion digit prime number is worth 250,000 footballs. Determined to get this prize, Quinn starts listing out every prime number he can think of. While doing so, he notices a pattern: for every integer $m \geq 2$, if there is no prime number p such that $p \leq \sqrt{m}$ and $p \mid m$, then m must be a prime. Help Quinn by proving that his pattern applies to all integers $m \geq 2$.

3. [10 pts] Yunha desperately wants to be drafted into the CIS 160 Staff Fantasy Football league team. In order to make it in, however, Rajiv tells her she must prove the following:

$$\sqrt{14} \text{ is irrational.}$$

Help Yunha prove the claim so her dreams can come true!

4. [10 pts] Hoping to win the CIS 160 Staff Fantasy Football league, best buds Andy and Zach decide to create a team together. Before finalizing the team, they each compile a shortlist of players that they would like to be considered for their team. Let A be the set of players Andy shortlisted and Z be the set of players Zach shortlisted. While they are trying to pick their final team, Andy suggests a compromise of choosing a team from the set $\mathcal{P}(A \cap Z)$. Zach passionately refuses, saying that it's unfair and instead suggests choosing a team from the set $\mathcal{P}(A) \cap \mathcal{P}(Z)$. Rajiv, who just so happens to overhear the conversation, shouts out "Stop arguing, they're the same thing!" Prove that:

$$\mathcal{P}(A \cap Z) = \mathcal{P}(A) \cap \mathcal{P}(Z)$$

Note that for the purpose of good practice, please strictly follow the standard set proof format for proving equality of two sets. i.e. you must prove $\mathcal{P}(A \cap Z) \subseteq \mathcal{P}(A) \cap \mathcal{P}(Z)$ and $\mathcal{P}(A) \cap \mathcal{P}(Z) \subseteq \mathcal{P}(A \cap Z)$. You must apply the basic definition of subset as done in lectures and recitations. Solutions that fail to follow the standard format will receive no point.

5. **[8 pts]** Given $n \in \mathbb{N}$, formulate a counting question whose answer is $(n^2)!/(n!)^n$. Justify your answer.

Do not provide an overly simple question as your solution. For example, we will not accept "Count the number of ways to choose one of $(n^2)!/(n!)^n$ items", as that does not demonstrate your understanding of the material.

6. **[8 pts]** As football fanatics, Anni and Anusha were granted special access to the Steelers practice where each of the 11 players on the field were wearing jerseys labeled 1 to 11. As the players walk out onto the field in a line, Anni realizes that if you subtract 1 from the jersey number of the first player, 2 from the jersey number of the second player, and so on so you subtract 11 from the jersey number of the eleventh player, the product of all of those values is an even number. As the players line up in random orders for their warm-up drills, Anusha realizes that Anni's statement is true for any order the players are lined up in. Prove that Anusha is correct.

7. **[12 pts]** With football season coming around, Arnold and Ranbir are talking about their favorite players. There are n players in the NFL from which Arnold and Ranbir each create a list of their favorites. There's just one problem: Arnold has been too busy grading CIS 160 homework and hasn't been able to keep up with the players this season! Afraid that he'll lose the illustrious moniker of "Football Head", Arnold decides his list must include every player on Ranbir's list. Now Arnold "Football Head" Jia is wondering the following:

(a) How many ways are there for Ranbir and Arnold to select players such that Arnold's list includes all the players on Ranbir's list?

(b) How many ways are there for Ranbir and Arnold to select players such that Arnold's list includes all the players in Ranbir's list and at least one other?

You can assume that the lists are unordered. Empty lists are allowed.