CIS 160 Assigned: November 5, 2020

Homework 10h

Due: 9:00 am EST, November 10, 2020

This assignment is due at the beginning of the class on the due date. Unless all problems carry equal weight, the point value of each problem is shown in [ ]. To receive full credit, all your answers should be carefully justified; in particular, please make sure to explicitly define your sample space for any probability question unless otherwise specified.

Please see Piazza for the clarification on collaboration policy @1716 @1750.

Also, please remember to double check that you have submitted the correct version of your homework onto Gradescope by re-downloading it.

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0. [2 pts] Mid-semester Feedback Form

Respond to the mid-semester feedback form Canvas quiz. We have changed the Canvas quiz settings so responses are anonymous, so please give us honest feedback so that we can improve the course moving forwards.

You can access the quiz here `https://canvas.upenn.edu/courses/1541967/quizzes/2487863` or on Canvas.

1. [10 pts] Preposterous Protein Predicament Prompts Peculiar Proposition Problem

Oliver is very particular about the milk he drinks and only likes Horizon Organic Whole High Vitamin D Milk. As such, he always keeps his fridge stocked with at least 7 gallons of milk. While buying his 7 gallons of Horizon Organic Whole High Vitamin D Milk for the week, Oliver's bag gives out from the weight of the milk, leaving Oliver stuck at the grocery store. Fortunately, Oliver happens to spot Saurabh passing by and asks for his help to tote his milk home. Saurabh agrees to help if and only if Oliver can solve the below question. Help Oliver get his milk!

Here are seven propositions:

$$x_1 \vee \overline{x_5} \vee x_7 \vee x_9$$
$$x_2 \vee x_6 \vee x_8 \vee \overline{x_1}$$
$$x_2 \vee \overline{x_4} \vee \overline{x_6} \vee x_8$$
$$\overline{x_4} \vee x_5 \vee x_7 \vee \overline{x_3}$$
$$\overline{x_6} \vee \overline{x_5} \vee x_8 \vee x_1$$
$$x_9 \vee \overline{x_8} \vee x_2 \vee \overline{x_1}$$
$$\overline{x_3} \vee x_9 \vee x_4 \vee x_2$$

Note that:

1. Each proposition is the logical or of four distinct terms of the form x_i or $\overline{x_i}$.

2. Only one of x_i or $\overline{x_i}$ can appear in any particular proposition.

3. No two propositions have the same combination of four terms.

Suppose that we assign true/false values to the variables $x_1, \dots, x_9$ independently and with probability $\frac{1}{2}$.

(a) What is the probability that the first proposition is true?

(b) What is the expected number of true propositions?

(c) Using only your answer to part (b), is it possible to show that there exists an assignment of the variables such that *all* of the propositions are true? Why or why not?

2. [10 pts] Partners in Cream

Adi's quest to create a high class milk store has led him into a tight connected web in the milk industry. In the milk industry every milk store owner can have partnerships with other owners (note that partnerships are mutual, and there can be at most one partnership between any two store owners). Store owners can ask for advice from any of their partners, but can also ask for advice from their partner's partners and so on. The milk industry prides themselves on cooperation and as such, claim that any store is able to ask advice from any other milk store. Given that every milk store has an even number of partners, prove that if any one pair of milk stores ends their partnership, the milk industry's claim still holds true. [1]

3. [10 pts] Crying Over Spilled Milk

Tomiwa has an infinite supply of almond milk and soy milk. He blindly reaches into his cabinet and pulls out a magical, infinitely deep carton of almond milk with a $\frac{4}{5}$ chance, and pulls out a magical, infinitely deep carton of soy milk with a $\frac{1}{5}$ chance.

Because Tomiwa can get kind of clumsy at times when pouring his milk, he has an independent $\frac{1}{10}$ chance of spilling milk every time he pours himself a glass of milk from the almond milk carton. He has an independent $\frac{1}{200}$ chance of spilling milk every time he pours himself a glass of milk from the soy milk carton.

Wanting his bones to grow big and strong, Tomiwa decides to drink as much milk as he can. He reaches into his cabinet and grabs either the almond milk carton or the soy milk carton, and repeatedly pours himself glasses of milk until he spills the milk. Not phased by the first spill, he continues to pour himself glasses of milk from the same carton until he spills the milk for the second time. After the second spill, Tomiwa gives up on his dream of growing big and strong

[1] We strongly encourage you to approach this problem without induction (our solution does not use induction). This is not to say it is not possible (correct proofs will receive credit), but induction proofs are not always so simple and it's important to learn how to construct graph proofs using non-inductive methods as well.

and proceeds to cry.

Let X be a random variable denoting the number of glasses of milk Tomiwa pours before his first spill, and let Y be a random variable denoting the number of glasses of milk Tomiwa pours between the first spill and the second spill. Determine whether X and Y are independent.

4. [10 pts] Lack Toes In Taller Ants

While going on a hike last Thursday, Weilin discovers a colony of n ants that had been exposed to mutant milk, causing them to grow extremely tall. She also notices that each of the n ants have a different number of toes. How strange! Intrigued, she decides to investigate further. Every day for the next n days, she picks an ant uniformly at random that she hasn't investigated before. Since she is slightly terrified of ants that lack toes (and for good reason), every time she investigates an ant a_j, she lets out a cry of anguish for each ant a_i that she has already investigated which had more toes than a_j. What is the expected total number of cries of anguish Weilin will have produced after she is done investigating the entire colony?

5. [10 pts] CIS 160 Cow-stume Party

Andy, wanting to show off his Milk-Jug costume, decides to throw a post-Halloween party for the CIS 160 staff. He wants to invite $n \geq 1$ TAs, and insists that each TA must compliment at least t other TAs on their costumes. In order to avoid insincere compliments, Andy also specifies that if Person A gives a compliment to Person B, then Person B can't give a compliment to Person A. Prove that Andy will need at least $2t$ TAs in order to throw a party that meets his requirements. Note that Andy himself does not participate in the complimenting.

6. [10 pts] Take One Down, Pass it Around, 18 Bottles of Milk on the Wall

To celebrate the deliciousness of dairy milk in comparison to alternative forms of "milk" such as almond and oat, Kara decides to create a display of 18 glass milk jugs in the milk room. 8 of the milk jugs have milk with 2% milkfat, and the other 10 are filled with whole milk. Kara decides to order the milk jugs uniformly at random on the table to make the display. In expectation, how many pairs of consecutive milks are there, such that one milk is 2% and the other is whole?

For example, if the milks are ordered WTTWWWWTTTTTWWWWWT (where W is whole milk and T is 2%) , we have 5 such pairs.

7. [10 pts] Hopping through Hallways and Hoping for Happiness (in the form of boba)

Ranbir's favorite boba store Milkyland is doing a special milk tea promotion. In the store, they've set up at least two DIY boba stations in different rooms. Each room has one boba station, and each boba station is reachable from another boba station either directly from one hallway or from a series of hallways that form a longer corridor. For their most loyal customers (like Ranbir), Milkyland let them in on a secret promotion: each hallway contains a filled out

Milkyland stamp card hidden among the wall decorations that can be removed and used to redeem a free milk-based drink.

To make sure he gets all the possible free drink stamp cards but not waste time, Ranbir wants to start from a boba station, walk through every hallway exactly once, and end at another boba station other than the one from which he started at. Prove that such a walk is possible if and only if there are exactly 2 DIY boba stations that have an odd number of hallways connecting them to other boba stations.