CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 9H

Assigned: October 28, 2021 **Due:** 8:30 AM ET, November 2, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, November 2, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:

- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may organize into collaboration teams of up to 3 current students. For each homework assignment, you can only be in one team and must list all team members on your homework submission using the provided LaTeX template, whether or not you specifically spoke with them. You may have different teams for different assignments. Collaboration must be strictly limited to discussion, and solutions must be written separately. For the complete collaboration policy, please consult the announcement on Piazza. Violations may seriously affect your grade in the course.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [14 pts] **A new challenger? Vote here!**

The Great Watermelon v. Mango War continues to wage on, when all of a sudden, a shocking third contender explodes onto the scene: the strawberry. As the war continues to grow more and more violent, the three tribes decide to temporarily declare a ceasefire and instead decide the winner through a bracketed tournament. The three tribes decide to send Watermelinda, Yuyango the Mango, and Strawb-Era as representatives.

The watermelon tribe proposes the following tournament structure. Watermelinda comes into the tournament as the first seed, which gives her the following advantage: Yuyango the Mango and Strawb-Era will play each other twice to begin the tournament. If one of them is able to defeat the other both times, they will move on to face Watermelinda in the finals. They then have two chances to defeat Watermelinda, and must defeat her twice in a row in order to win the tournament. Note that if Watermelinda wins round 3 of the tournament, there will be no fourth game played. If Yuyango the Mango and Strawb-Era split their two game series, then Watermelinda will automatically be crowned the champion.

You are an analyst for the mango tribe. You're given the following data accumulated from the Great Watermelon v. Mango War:

- The probability that Yuyango the Mango defeats Strawb-Era in a single match is 0.7.
- The probability that Watermelinda defeats Yuyango the Mango in a single match is 0.5.
- The probability that Watermelinda defeats Strawb-Era in a single match is 0.6.

Assume that all matches are independent and that it's impossible to tie.

(a) Given that no matches have been played yet, determine the probabilities that:

   i. Watermelinda will win the tournament without playing a single match.

   ii. Yuyango the Mango will play Watermelinda in the final round.

   iii. Watermelinda is crowned the champion of the entire tournament.

(b) Given that Watermelinda played at least one match, determine the probabilities that:

   i. Yuyango the Mango is playing Watermelinda in the final round.

   ii. Watermelinda wins the overall tournament.

(c) Given that Watermelinda played exactly one match, what is the conditional probability that Yuyango the Mango defeated Strawb-Era in both of their matches?

2. [12 pts] Jay the Watermelon Weightlifter

Jay has been craving watermelon for months, and he finds out that Costco is having a sale. He purchases 2^{n+1} watermelons in total, $n \in \mathbb{N}$, where each watermelon has an arbitrary natural number weight. The weights are not necessarily distinct. However, as watermelons are not the easiest fruit to carry around (as opposed to, oh I don't know, a reasonably sized fruit like a mango) Jay can only carry 2^n of them back to his dorm. Prove that, for every possible weight configuration of 2^{n+1} watermelons, it is possible for Jay to select and bring 2^n watermelons back such that the total weight of the watermelons are divisible by 2^n.

3. [12 pts] MUKIL'S MANGO MAYHEM

Mukil wants to collect mangoes from a collection of mango shops for his annual spree he likes to call Mukil's Mango Mayhem. Between some pairs of mango shops, there is a navigable bidirectional road. Between any two mango shops, there is at most one road. Mukil wants to partition the mango shops into two non-empty sets: a set S containing shops with sweet mangoes and W containing shops with sour mangoes. Mukil notices that no matter how he partitions his shops, there is always a road between some pair of shops a and b such that $a \in S$ and $b \in W$. Prove that this happens if and only if Mukil can travel along some roads to get from any mango shop to any other mango shop.

4. [12 pts] "mAnGoS wIn" (this is the top of a water bottle)

The CIS160 TAs are in a heated argument about which fruit is the GOAT (greatest of all time). The argument has reduced to two fruits: watermelons versus mangoes. In a conquest to decide the debate, they fight it out to a battle of the best involving fights of various skills: Capture the Flag, rocks-papers-scissors, and mental arithmetic. Mangos win. In light of goods sportsmanship, the TAs take turns shaking hands with each other. However, it's possible that some TAs never get to shake hands.

To document their experience, Quinn records each pair of TAs who shake hands. When he's finished, he realizes that he has formed a simple, undirected graph!

He calls the resulting graph G, and to make sure that everyone has shaken hands with each other, Quinn sets out to create $\overline{G}$, the graph representing pairs of TAs who haven't shaken hands yet. More formally, consider the initial graph with the set of TAs V and the set of handshakes H such that $G = (V, H)$. Define $\overline{G} = (V, \overline{H})$, where

$$\overline{H} = \{\{x, y\} \mid x \neq y, \{x, y\} \notin H\}$$

Define a *cut vertex* to be a vertex such that the subgraph obtained by deleting it and all of its incident edges has more connected components than the original graph.

Prove that if a TA's vertex v is a cut vertex in G, then v is not a cut vertex in $\overline{G}$.

5. [8 pts] Jasmine + Cats 4ever <3

A "friend" of Jasmine's decided to gift her a bag of mangoes, forgetting that she is very allergic to mangoes. Unwilling to let the manGO (i.e. mango go) to waste, she decided to lay them out across the room and join them with strings to create a maze for her cats.

After finishing the maze, she realised that the drawing looks like a tree. In fact, it is a connected, acyclic graph! She likes it and decides to call it T.

Jasmine notices that in T, which has at least two vertices, each vertex that is adjacent to a leaf has a degree of at least 3. Prove that there exist two distinct leaves in T, x and y, such that $N(x) = N(y)$ (where $N(v)$ is the set of neighbors of a vertex v).

6. [12 pts] The Legend of the Kommons King (he who decides when we get tater tots)

In an effort to make the fruit bar at Kommons more appealing, Krazy Kyle decides to add pieces of mango to the watermelon cups. Kyle has w watermelon cups, $w \in \mathbb{Z}, w \geq 2$, and $2^m, m \in \mathbb{N}$, pieces of mango scattered among them (where each mango piece is only added to one watermelon cup). However, the Kommons King decides that he would rather have a single watermelon cup with all of the pieces of mango as a main attraction. Unfortunately, Kyle is only able to move mango pieces in the following way: Pick two distinct watermelon cups, say cup A with p mango pieces and cup B with q mango pieces, where $p \geq q$; Kyle can then move exactly q mango pieces from cup A to cup B. Prove that it is possible for Kyle to move all of his mango pieces to one watermelon cup!