CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 6T

Assigned: October 5, 2021 **Due:** 8:30 AM ET, October 7, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, October 7, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:
- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may not collaborate with anyone via any means.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources. **If you use the multiplication rule on a question in this homework, you must explicitly cite the multiplication rule.**

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [10 pts] Fellas, is it animal abuse for Joseph to starve his vegetarian dinosaurs? Answer the following questions. You may express your answer as a percentage, decimal or fraction. **For this question only, you do not need to show your work. Only your final answer will be graded.**

For questions (a), (b), (c), find the probabilities of the events described based on the following experiment: A field contains n trees labeled $1, 2, \ldots, n$. A group of (vegetarian) dinosaurs choose a tree to visit uniformly at random, they eat lunch there, and they go home. Then for dinner, the dinosaurs go back to the same field, choose a tree to visit uniformly at random, eat dinner there, and go home. Note that they can visit the same tree twice if they so choose. Question (d) is based on a similar experiment (read the problem statement below for details).

For questions (e), (f), (g), find the probabilities of the events described based on the experiment of rolling two standard fair six-sided dice.

Statements (h), (i), (j) are based on the following scenario: Joseph's Jurassic Park consists of two types of dinosaurs: raptors and triceratops. Due to the declining tree population, 61% of his raptors and 89% of his triceratops suddenly die of hunger. State whether each claim is either **true**, **false**, or **can't be decided** based on the information provided.

(a) The first tree chosen is number 1 and the second tree chosen is number n.

(b) The numbers of the two trees are consecutive integers with the first tree's number being one less than the second tree's number.

(c) The second tree chosen has a higher number than the first tree chosen.

(d) The dinosaurs realize that in the experiment, they might end up at the same tree for lunch and dinner and they decide that they must avoid that possibility. Now, recalculate (a), (b), and (c), assuming this time that the dinosaurs do not even consider going to the same tree for both meals, meaning that the number for the first tree cannot even be considered while choosing the second tree.

(e) The maximum of the two numbers rolled is less than or equal to 3.

(f) The maximum of two numbers rolled is exactly equal to 3.

(g) Recalculate parts (e), (f) for x instead of 3, where $1 \leq x \leq 6$. Give a solution in terms of x, instead of a piece wise solution.

(h) The overall proportion of Joseph's dinosaurs that died is exactly 75%.

(i) The overall proportion of Joseph's dinosaurs that died is between 61% and 89%.

(j) The statement in part (h) is true if and only if the raptors to triceratops ratio prior to deaths at Joseph's Jurassic Park is 1-1.

2. [10 pts] I also hate it when I go through a magical time portal in Towne which is then blocked by an induction problem

On his way to recitation, Andrew discovers a magical time portal in a corner of Towne Hall. He steps inside and finds himself in the middle of the Mesozoic Era, staring straight at the toes of a Dreadnoughtus. Scared out of his wits, Andrew turns around to go back immediately but finds that the portal is now blocked by an induction problem:

Suppose that the sequence $a_1, a_2, \ldots$ is defined by $a_1 = 1$ and $a_{n+1} = 3a_n^2$, for each integer $n \geq 1$. Prove using induction that $a_n \leq 3^{2^n}$, for all positive integers n.

(a) In a hurry to unlock the portal to make it back to his recitation intact, Andrew first lets his induction hypothesis assume $a_k \leq 3^{2^k}$, for some positive integer k. Does this work? If not, show why.

(b) Then, Andrew realizes that he could try to prove the statement $a_n \leq 3^{(2^n-1)}$ using induction. Prove Andrew's claim and show that it implies what he wanted to prove in part a.

3. [10 pts] "How dareth thou besmirch my good name?"

Krish the Kritosaurus, a dino known for his stunning proofs and ingenious techniques, has had his name besmirched by Helen the Hadrosaurus who claims that not all of his so-called flawless proofs are actually flawless. For dinosaurs, perfect proofs remain the highest source of pride, so this is a serious accusation. To help discover the truth behind this debacle, for each of the following, indicate whether or not the proof is flawed. If the proof is incorrect, clearly explain where the error in the proof lies. If you say that the proof is correct, you do not need to explain further.

(a) **Claim:** $\forall n \in \mathbb{Z}^+$, if p and q are positive integers such that $\max(p, q) = n$, then $p = q$.
Proof: Base Case: Let $n = 1$. If p and q are positive integers such that $\max(p, q) = 1$, then p and q must both be 1, satisfying the claim.

Induction Hypothesis: Assume that the statement holds for $n = k$, where k is an arbitrary positive integer. In other words, we assume that if p and q are positive integers such that $\max(p, q) = k$, then $p = q$, for some positive integer k.

Induction Step: Consider the case where $n = k + 1$, and let p' and q' denote two positive integers such that $\max(p', q') = k + 1$. Then we must have $\max(p' - 1, q' - 1) = k$, which, by our induction hypothesis, implies that $p' - 1 = q' - 1$, and hence that $p' = q'$. This

proves our claim.

(b) **Claim:** For any positive real number x, if x is irrational, then $2010x$ is irrational.

Proof: Suppose $2010x$ is rational. Then $2010x = p/q$, for some integers p and q, with $q \neq 0$. Therefore, $x = p/2010q$, where p and $2010q$ are integers with $2010q \neq 0$, so x is rational. Therefore we have proven the contrapositive – if $2010x$ is rational, then x is rational and hence the claim is true.