CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 5T

Assigned: September 28, 2021 **Due:** 8:30 AM ET, September 30, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, September 30, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:
- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may not collaborate with anyone via any means.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources. **If you use the multiplication rule on a question in this homework, you must explicitly cite the multiplication rule.**

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [6 pts] Life If All Natural Numbers Were Divisible By 4 >>>

During office hours, Christian presents the following proof.

Statement. All natural numbers are divisible by 4.

> **Proof.** Suppose, for the sake of contradiction, the statement is false. Let X be the set of counterexamples, i.e. $X = \{x \in \mathbb{N} \mid x \text{ is not divisible by } 4\}$. The supposition that the statement is false means that $X \neq \emptyset$. Since X is a non-empty set of natural numbers, it contains a least element x.
>
> Note that $0 \notin X$ because 0 is divisible by 4. So $x \neq 0$. Now consider $x - 4$. Since $x - 4 < x$, it is not a counterexample to the statement. Therefore $x - 4$ is divisible by 4; that is, there is an integer a such that $x - 4 = 4a$. So $x = 4a + 4 = 4(a + 1)$ and x is divisible by 4, contradicting $x \in X$.

Hoot the Hadrosaurus, a confused student, points out that Christian must be wrong, since not all natural numbers are divisible by 4. Christian refuses to budge and claims that his proof clearly demonstrates that all natural numbers are divisible by 4.

Find and explain where Christian goes wrong in his proof.

2. [10 pts] A Fibonacci Countdown Adds Suspense Because You Say 1 Twice!

As an asteroid approaches Earth, dinosaur Winnie and dinosaur Weilin decide that they want to enjoy their last moments on Earth together by discussing their favorite thing in the world: Fibonacci numbers! Dinosaur Winnie made a claim about Fibonacci numbers, but dinosaur Weilin is unsure if Winnie is correct! Help Weilin prove Winnie's claim before the asteroid reaches Earth! Fibonacci numbers are denoted as F_i where $F_1 = F_2 = 1$ and $F_{j+2} = F_j + F_{j+1}$. Help Weilin prove that for all integers $n \geq 1$,

$$\sum_{i=1}^{n} F_i^2 = F_n F_{n+1}$$

3. [14 pts] Dino Etiquette 101: It's OK To Ask For Some-dino's Age

After the asteroid hits, Richard the dinosaur stumbles out of the rubble, realizing that he somehow miraculously survived. As Richard gets used to his surroundings, he soon realizes that he doesn't recognize any of the dinosaurs around him. Because of this, Richard sadly plops on the ground, saying, "All my friends are dead :(".

Luckily for Richard, there are n other surviving dinosaurs around him that Richard can be new friends with! As part of their introductory efforts, Richard decides to write out the ages of all of his n new friends (note that a dinosaur can be exactly zero years old). Because he likes precision, Richard decides to write every single dinosaur's exact age to the decimal; this means

that each of these dinosaur's ages aren't necessarily just integers, but instead are nonnegative real numbers. Richard maps out each of the n dinosaur's ages to a specific variable x_i, where for positive integers $i \leq n$, x_i corresponds to the exact age of his i^{th} new friend. Richard calculates the average age of his n friends to be $\overline{x} = \frac{(x_1 + x_2 + \cdots + x_n)}{n}$. After computing this average, Richard makes a few observations, but he isn't sure if they're necessarily true.

(a) Prove that at least one of Richard's n new friends is at least as old as $\overline{x}$.

(b) Prove that strictly fewer than half of Richard's n new friends are older than $2\overline{x}$.