CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 5H

Assigned: September 30, 2021 **Due:** 8:30 AM ET, October 5, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, October 5, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:

- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may not collaborate with anyone via any means.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources. **If you use the multiplication rule on a question in this homework, you must explicitly cite the multiplication rule.**

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [9 pts] Velociraptors were carnivores, but Velo-sara-ptors are herbivores

Era the Tric-Era-tops and Sara the Velo-Sara-ptor pride themselves on being vegetarian. However, just because they are plant lovers does not mean they are less aggressive than their carnivorous counterparts. In fact, to fuel their thirst for competition, Era and Sara are having a competition to see who can eat the most leaves. For a given positive integer of n, Era can eat 2^n leaves, whereas Sara can eat n^3 leaves. Sara observes that for all integers $n \geq 10$, Era will be able to eat at least as many, if not more, leaves than herself. Can you prove that Sara's observation is correct?

2. [12 pts] Proofs we'd like you to think about

For each of the "proofs" below, say whether the proof is valid or invalid. If it is invalid indicate clearly as to where the logical error in the proof lies and justify why this is a logical error. If the proof is valid, you can simply say so. Just stating that the claim is false will not be awarded credit.

(a) **Claim:** $\forall n \in \mathbb{Z}^+, n^2 \leq n$.

Proof:

Base Case: For $n = 1$, the claim is true since $1^2 = 1$.

Induction Hypothesis: Assume that $k^2 \leq k$, for some $k \in \mathbb{Z}^+$.

Induction Step: We need to show that

$$(k+1)^2 \leq k+1$$

We can see that

$$k^2 \leq k^2 + 2k = (k^2 + 2k + 1) - 1 = (k+1)^2 - 1$$

Since $(k+1)^2 \leq k+1$,

$$(k+1)^2 - 1 \leq (k+1) - 1 = k$$

Thus we get $k^2 \leq k$, which we know is true by the induction hypothesis.

(b) **Claim:** $\forall n \in \mathbb{Z}^+, 5^n = 5$.

Proof: We will prove the claim using strong induction on n.

Base Case: For $n = 1$, the claim is true since $5^1 = 5$.

Induction Hypothesis: Assume that $5^j = 5$, for all integers j s.t. $1 \leq j \leq k$ for some $k \in \mathbb{Z}^+$.

Induction Step: We need to prove that $5^{k+1} = 5$, and we have

$$\begin{aligned} 5^{k+1} &= \frac{5^k 5^k}{5^{k-1}} \\ &= \frac{5 \cdot 5}{5} \quad \text{(using induction hypothesis)} \\ &= 5 \end{aligned}$$

This completes the strong induction proof, so $5^n = 5$, for all $n \in \mathbb{Z}^+$.

(c) **Claim:** For all negative integers n,

$$(-2) + (-4) + \ldots + (2n) = -n^2 + n$$

Proof: We will prove the claim using induction on n.

Base Case: The claim holds when $n = -1$ since $-2 = -(-1)^2 + (-1) = -1 - 1 = -2$.

Induction Hypothesis: Assume that $(-2) + (-4) + \ldots + (2k) = -k^2 + k$, for some $k \in \mathbb{Z}$, $k \leq -1$.

Induction Step: We want to prove that the claim is true when $n = k - 1$. That is, we want to prove that

$$(-2) + (-4) + \ldots + (2k) + (2(k-1)) = -(k-1)^2 + (k-1)$$

$$\begin{aligned} \text{L.H.S.} &= (-2) + (-4) + \ldots + (2k) + (2(k-1)) \\ &= (-2) + (-4) + \ldots + (2k) + (2k-2) \\ &= -k^2 + k + (2k-2) \qquad \text{(using induction hypothesis)} \\ &= -k^2 + 3k - 2 \\ &= (-k^2 + 2k - 1) + (k-1) \\ &= -(k-1)^2 + (k-1) \end{aligned}$$

This completes the induction proof.

(d) **Claim:** $\forall n \in \mathbb{N}$, $5n = 0$.

Proof: We will prove the claim using strong induction on n.

Base Case: The claim holds when $n = 0$ since $5 \cdot 0 = 0$.

Induction Hypothesis: Assume that $5j = 0$, for all $0 \leq j \leq k$, for some $k \in \mathbb{N}$, $j \in \mathbb{Z}$.

Induction Step: We must show that $5(k+1) = 0$. Let $k + 1 = a + b$, where $a > 0$ and $0 < b \leq k$ are integers. From the induction hypothesis we know that $5a = 0$ and $5b = 0$, therefore

$$5(k+1) = 5(a+b) = 5a + 5b = 0 + 0 = 0$$

This completes the induction proof.

3. [9 pts] Generic Question Name

Prove using induction that for all positive integers n, and for any integers a and b with $a \neq b$, $a^n - b^n$ is divisible by $a - b$

4. [10 pts] This is because TAs are not humans

During the deep tranquil hours of the twilight, an unknown time-magician lands on Earth and yells, "To the Mesozoic Era!". Every human on earth vanishes except a group of CIS 160 TAs. They are surrounded by thousands of dinosaurs looking to hunt upon them. Luckily, the time-magician wants to be entertained by the TAs so he provides them a chance to survive. There are $n \geq 2$, $n \in \mathbb{Z}$ dinosaur-proof shelters scattered across the lands. Between *every* two shelters A and B, there is an underground track for trolleys. For any given track between shelters X and Y, trolleys can go only from X to Y, only from Y to X, or in both directions.

Linda, who will coordinate the travel, notices a problem: some shelters might be dead-ends, meaning they have no departing trolley carts! Help Linda investigate this issue.

The all-knowing Tien appears behind you. You are allowed to ask Tien at most $2(n-1)$ questions of the form "Can a TA at shelter A take a single trolley cart to shelter B?". He will truthfully respond to each question with "Yes" or "No". Prove, using induction, that if there are n shelters, we can find a dead-end shelter, if one exists. (*Hint:* First, consider how many dead end shelters there could be)

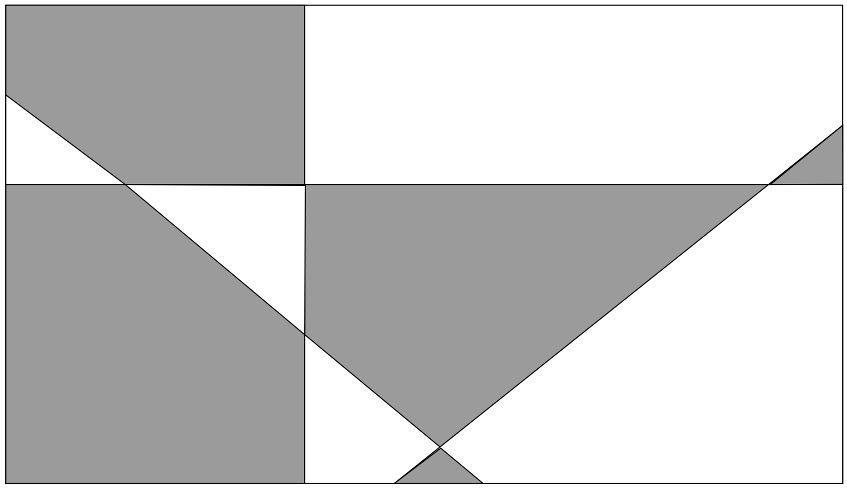
5. [10 pts] Terribly Territorial Tribes

Thanks to climate change, the resources on the piece of land that the dinos love to frequent are becoming more and more scarce. Naturally, arguments arise and the dinos want to claim pieces of the land as their territory.

In order to prevent fights, the landlord, Bethany decided to split the land. The land can be represented as a rectangle. Bethany draws $n \geq 0$ straight lines to divide the land, where $n \in \mathbb{Z}$. Each line starts on one edge of the rectangle and ends on another edge. No two lines have the same two endpoints. The lines divide the land into polygon regions that are not necessarily rectangular.

There are two main tribes of dinos, one led by Richard and the other led by David. To prevent tribe members from uniting with one another and cause fights with the other tribe, Bethany needs to make sure that each dino can get their own piece of land, but no two dinos from the same tribe would share adjacent pieces of land. Two pieces of land are defined as adjacent if they share a common boundary line. Note that regions that touch each other only at their corners can have the same color.

Prove that Bethany is able to split the land given the conditions above, no matter how many lines she draws on the land.



An example of the land with $n = 4$ lines

6. [10 pts] Please Click Me

To compete against the domination of humanity by AI-based robots, Mukil the mad scientist begins to create his own army of artificially made dinosaurs. Mukil believes that a very specific number of horns will allow him to exploit the robots' greatest weakness. Each dinosaur has a different number of horns, ranging from 1 to $2n$, $n \in \mathbb{Z}^+$. To find what number of horns would be best, he randomly chooses $n+1$ unique dinosaurs from this group to battle the robots. Prove that in any assortment of dinosaurs that could be selected, there will always be two dinosaurs whose number of horns are relatively prime.

7. [10 pts] Claire the Caretaker

After a long period of introspection, Claire has become a very nice and kind person. In an attempt to repent for her previous sins, she is now the caretaker of a new generation of dinosaurs. She counts the number of dinosaurs she is raising and realizes that she has $101^n - 1$ dinosaurs for some $n \in \mathbb{Z}^+$. Help her prove using the Pigeonhole Principle that, for some $n \in \mathbb{Z}^+$, the number of dinosaurs she is raising is divisible by Claire's favorite age, 19. (*Hint:* Consider looking at numbers of the form 101^k)