CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 4H

Assigned: September 23, 2021 **Due:** 8:30 AM ET, September 28, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, September 28, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:
- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may not collaborate with anyone via any means.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources. **If you use the multiplication rule on a question in this homework, you must explicitly cite the multiplication rule.**

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [18 pts] There's always a (fall) back

Will, an avid leaf cruncher, has finally finished his induction homework! Or so he thinks. To his dismay, after turning the paper over there were actually two more induction problems on the back! Help Will prove the following two statements using induction so that Will can finally go see the pretty leaves on Locust.

(a) Prove that for all $n \in \mathbb{Z}^+$,

$$\sum_{i=1}^{n} i^3 = \left(\sum_{i=1}^{n} i\right)^2$$

(b) Prove that for all integers $n > 1$,

$$1 + \frac{1}{4} + \frac{1}{9} + \cdots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

2. [16 pts] No Missed Stakes

Ishaan is in charge of the CIS 160 farm which has 38 distinguishable gardens, numbered from 1-38. He is trying to deal with the lantern fly infestation that might ruin the fall harvest. Ishaan decides the best way to deal with lantern flies is to build scarecrows in each of the 38 distinguishable gardens. There are three steps to building a scarecrow: planting a stake, clumping a bale of hay onto the stake, and putting clothes on the scarecrow. Note that these steps must be performed in chronological order. Ishaan has numbered stakes 1-38, bales of hay 1-38, and clothes 1-38 for each of the 38 specific gardens.

(a) Ishaan is trying to plan out his schedule to assemble the 38 distinguishable scarecrows. He wants to make sure that each numbered objects is matched with its correspondingly numbered garden. Note that Ishaan doesn't have to construct a single scarecrow all at once, and can spread out the steps provided that the steps happen in the correct order for any single scarecrow. How many ways are there for Ishaan to schedule out how to assemble the 38 scarecrows?

(b) Ishaan is trying to plan out his schedule to assemble the 38 distinguishable scarecrows except this time, he decides that any garden can get any number stake, hay bale, and clothes. Note that once again, Ishaan doesn't have to construct a single scarecrow all at once, and can spread out the steps provided that the steps happen in the correct order for any single scarecrow. How many ways are there for Ishaan to schedule out how to assemble the 38 scarecrows?

3. [14 pts] The Best Time to Wear a Striped Sweater

Ria is a sweater weather fanatic and has been collecting copies of this one specific coziest sweater for the fall since she was young. Unfortunately with global warming, it's currently just not sweater weather right now :(, so Ria is being forced to clean out her horde of sweaters. She has 32 identical copies of this cozy sweater to give away to 5 of her TA friends. She specifically wants four of the friends to receive an odd number of sweaters and the last person should receive a number of sweaters that yields a remainder of 2 when divided by 3. How many ways can Ria (very sadly) give away her sweaters to her friends?

4. [12 pts] University of Pennsylvania School of Engineering and Applied Science, Computer and Information Science 160: Mathematical Foundations of Computer Science, Fall 2021, Homework 4H, Question #4 Title

Show that the number of r-combinations of the multiset $M = \{1 \cdot a_1, \infty \cdot a_2, \ldots, \infty \cdot a_k\}$ is given by

$$\binom{k+r-3}{r-1} + \binom{k+r-2}{r}$$

where $k > 1, r \geq 1$ and $k, r \in \mathbb{Z}$. Recall that an r-combination of the multiset M is an unordered collection of r elements from M. Your argument should not involve any kind of algebraic manipulation of the expression.

5. [10 pts] Carrotorial Proof Time

David and Brandon are having a carrot eating contest. As they crunch on the carrots, they each count their progress using mathematical operations and positive integers n and k, where $n \geq k$. When the timer rings, David announces that he finished $k\binom{n}{k}$ carrots while Brandon finished $n\binom{n-1}{k-1}$. As the judge, Christian points out that they both ate the same amount of carrots. Help explain why they tied by giving a combinatorial proof for the following identity:

$$k\binom{n}{k} = n\binom{n-1}{k-1}$$

where $k, n \in \mathbb{N}, 1 \leq k \leq n$.