CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 3H

Assigned: September 16, 2021 **Due:** 8:30 AM ET, September 21, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, September 21, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:
- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may not collaborate with anyone via any means.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources. **If you use the multiplication rule on a question in this homework, you must explicitly cite the multiplication rule.**

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [10 pts] joseph is so athletic and fit i wish i could be like him

(a) Joseph the otter aspires to become an Olympic swimmer so he has to start training hard for the Paris 2024 Olympics. He decides to build a personal swimming pool in his New York mansion. The swimming pool is in the shape of a cube with integer side lengths. He noticed that, "if the volume of the swimming pool is an even integer, the side length of the swimming pool would be too!" Can you help Joseph prove the claim? Recall that the volume of a cube with side length s is s^3.

(b) After training in solitude for a month, Joseph realized that he needs friends. He began his construction work by first creating some swimming pool covers.

Joseph has a large piece of fabric that is a ft^2, which he will cut into b equal-area pieces to cover the b miniature swimming pools, each with area $\sqrt[3]{100}$ ft^2, where a and b are positive integers. All fabric must be used and the pieces need to have the exact same area.

While trying to figure out how to cut the fabric, Joseph fell asleep and had a nightmare, where Yuyang the Big Brain God appeared and said, "your process will never produce a piece of your desired size. You need to find better ways to make friends!" Is Yuyang correct? Prove your answer.

2. [8 pts] Sara & Sneha's Sea Searching Spree

As part of their Oceanology class, Sara and Sneha are on a quest to observe sea otters in the wild. They take a walk along the seaside in hopes of crossing paths with a sea otter. Sara decides to model the amount of time they must wait until they see a certain number of sea otters using the equation $t^2 + t + 1 = n^2$, where t is the amount of time in minutes and n is the number of sea otters. However, Sneha immediately spots a flaw in Sara's equation, telling her that she needs to use a different model since $t^2 + t + 1 = n^2$ has no positive integer solutions (where both t and n are positive integers), which is an issue since amount of time in minutes and number of sea otters must both be positive integers. Prove that Sneha is correct.

3. [10 pts] Are you smarter than a hairy otter?

The Secret Association of Sea Otters (S.A.S.O.) is highly grateful to Era Dewan for proposing a sea otter theme for CIS 160 Fall 2021 Homework 3H. In order to thank her, they brought her to their secret base in Otter Space, where their top mathemagician, Hairy Otter, gave her a request of utmost importance. In order to save the sea otters, she was required to prove the following: if $\mathcal{P}(A) \cup \mathcal{P}(B) = \mathcal{P}(A \cup B)$, then $A \subseteq B$ or $B \subseteq A$. Help Era Dewan prove her claim and save the sea otters!

4. [8 pts] Boring problem, no story. Is the number even or odd(er)?

Let $x_1, x_2, \ldots, x_{2021}$ be a permutation of the numbers from 1 to 2021. Prove or disprove that the product

$$(x_1 - 1)(x_2 - 2)\cdots(x_{2020} - 2020)(x_{2021} - 2021)$$

must be an even number.

5. [8 pts] No, otters don't actually build dams :(

Kyle the otter is trying his best to build a dam in the Schuylkill River. In preparation for his arduous task, Kyle has managed to prepare 864 completely unique sticks (naturally, these are incredibly large sticks so that Kyle can effectively dam the entire river). Kyle, being the genius engineer that he is, has thought up of a construction made out of exactly 432 wooden X's. In order to create these wooden X's, Kyle has to create pairs of his 864 distinct wooden sticks (Note that the order of the sticks within the pair does not matter, and that the order of the pairs doesn't matter either). How many ways can Kyle pair his sticks to create all the X's that he needs?

6. [10 pts] sad otter-watcher hours

Richard Chai's favorite hobby is holding hands, but because of the pandemic, he hasn't been able to hold hands with as many people as he would like (germs, ya know?). To help with his hand holding withdrawal symptoms, he decides to go to the zoo to watch the sea otters hold hands.

Richard observes $2n$ distinct otters swimming around, where $n \in \mathbb{N}$, $n \geq 2$. Each otter is holding hands with one other distinct otter, where hand holdings are mutual. After watching the otters for half a day or so, Richard starts to feel left out. He breaks into the exhibit and makes the otters let go of their partners' hands and get into a circular formation. Richard then remembers his second favorite hobby - solving combinatorics problems. He decides to count how many ways the $2n$ otters could arrange themselves in a circle. However, Richard has been focusing too much on hand holding lately and has gotten a bit rusty at combinatorics. He needs help figuring out the number of possible arrangements.

Note that two arrangements are considered the same if, for each otter, the otter to his or her left is the same in both arrangements. Suppose that in the otter circle, k out of the $2n$ otters where $k \leq n$ demand to be next to the otter they were just holding hands with. (Note that no two of the k demanding otters were previously holding hands with each other). The remaining otters may be next to their hand holding partners, but do not require it. How many ways are there to arrange the $2n$ otters in a circle such that the k demanding otters get to be next to the one they were holding hands with?

7. [8 pts] If you say "kelp" over and over, it sounds weird

The sea otters Ananya, Bethany, and Claire are competing to collect the most giant kelp. Ananya has A blades of kelp, Bethany has B blades of kelp, and Claire has C blades of kelp. Otter Olivia, the reigning kelp collection champion, notices that $A^2 + B^2 = C^2$. Otter Olivia hypothesizes that A, B, and C cannot all be prime numbers. Prove that A, B, and C cannot all be prime numbers.

8. [8 pts] Mukil is the cutest (both sea otter and human form <3)

Mukil, the sea otter, plans on proposing to a sea otter he met during his trips to the Pacific Ocean. He is nervous. Reflecting on his past mistakes, he wants to arrange a garden of flowers for her. He arranges the pot of flowers such that each pot forms the corners of a convex polygon with n sides, $n \in \mathbb{N}, n \geq 4$. Recall that a convex shape is one where all diagonals lie entirely within the polygon. Then, he takes petals of minuscule roses and lays them down in a line between every non-adjacent pair of pots of flowers. After all lines have been laid down between non-adjacent pairs, he wonders how many intersections the rose lines have at interior points. Can you help Mukil figure out how many such intersections exist? You can assume that no more than two lines intersect at the same point.