CIS 160 — Mathematical Foundations of Computer Science
Homework Assignment 11H

Assigned: November 11, 2021 **Due:** 8:30 AM ET, November 16, 2021

This homework is due electronically on Gradescope at 8:30 AM ET, November 16, 2021. To receive full credit all your answers should be carefully justified.

Please make note of the following:

A. Standard Deductions:

- 5 points will be deducted from your homework if you do not use the provided LaTeX template.
- 5 points will be deducted from your homework if you do not select pages when submitting to Gradescope.
- No credit will be awarded to assignments that are not typeset in LaTeX.

B. Solutions: Please make sure to keep your solutions clear and precise. While no points will be deducted for overly verbose solutions, clarity and brevity are important skills that can be developed through CIS 160. If multiple solutions are given, only the first one will be graded. *Solutions must be given in closed form (as defined on Piazza).*

C. Collaboration: You may organize into collaboration teams of up to 3 current students. For each homework assignment, you can only be in one team and must list all team members on your homework submission using the provided LaTeX template, whether or not you specifically spoke with them. You may have different teams for different assignments. Collaboration must be strictly limited to discussion, and solutions must be written separately. For the complete collaboration policy, please consult the announcement on Piazza. Violations may seriously affect your grade in the course.

D. Citations: All solutions must be written in your own words. If you would like to use part of a solution from a problem presented in lecture, recitation, or past homework solutions you may do so with attribution; i.e., provided you add a comment in which you make clear you copied it from these sources.

E. Outside Resources: Any usage of resources outside of the course materials on the course website or Canvas is strictly prohibited. Violations may seriously affect your grade in the course.

F. Late Policy: We will allow you to drop two homework assignments assigned on a Tuesday and two homework assignments due on a Thursday (i.e. two 'T' homeworks and two 'H' homeworks). Because of this, we will not accept late homework under any circumstances. If you will be missing school for an extended period of time due to severe illness, please notify the professor.

1. [14 pts] Nobody is as excited for Taylor Swift homework as Bethany!

Bethany, a great dancer and an even better singer, is known among 160 staff for being able to sing the most Taylor Swift songs in one day. However, the number of songs she can sing per day fluctuates every day for n days, $n \geq 3$. Particularly, on the ith day for $1 \leq i \leq n$, she can sing S_i songs in that day, where S_i is an integer chosen uniformly at random between 1 and V inclusively. Furthermore, the number of Taylor Swift songs she can sing per day is never the same for any two days and $V > n$. To save her vocal cords, Bethany strategically chooses to only perform on "performance days". The ith day is a performance day if $S_{i-1} < S_i$ and $S_i > S_{i+1}$ for $1 < i < n$. The first day is a performance day if $S_1 > S_2$ and the last day is a performance day if $S_n > S_{n-1}$. Help Bethany calculate the expected number of days she will perform.

2. [14 pts] Taylor's Bangers

Taylor Swift writes bangers. She knows that 90% of the songs in her latest album, *Red (Taylor's Version)*, are bangers. That is, 90% of her songs have a 1/20 chance of topping the charts on any given day, independent of their chart status from the previous day(s). The remaining 10% of songs on the album (non-bangers) are not fully appreciated by the public and have a 1/1000 chance of topping the charts on any given day, also independent of their chart status from the previous day(s).

Taylor wants to drop a song early to hype up her Swifties, but she is unable to differentiate between bangers and non-bangers. She randomly chooses a song from *Red (Taylor's Version)* and releases it to the public. Let X be the random variable denoting the number of days until this song tops the charts.

Years later, Taylor decides to work on another project: *Red (Taylor's Newer Version)*, rereleasing the same set of songs. In an effort to build hype for this new album's release, she drops the *same* song early again from this new album. Assume that this song has the same probability of topping the charts on any given day as the corresponding song from *Red (Taylor's Version)*. Let Y be the random variable denoting the number of days until this "new" song tops the charts.

Are the random variables X and Y independent?

For this question, you do not need to define a sample space, but you must define any other events or random variables you may use.

3. [15 pts] Magic, madness, heaven, sin

Taylor's got a long list of ex-lovers. She represents each ex-lover with a heart. Her collection of hearts are connected by strings, where each string starts at one heart and ends at another.

She notices that no two strings start and end at the same heart and there is at most one string between any two hearts.

Swift notices that these hearts and strings form loops. She recognizes these loops by starting from a heart, taking strings to other hearts, not repeating hearts along the way, and ending at the original heart. As she looks at each loop, she realizes that they all have an odd number of strings.

She decides to color the strings along each loop with different shades of red, so that she can distinguish between loops. She colors the strings in each loop with the same color, and each loop has a different color. Taylor claims that she won't have to color any string with more than one color because all of the loops have odd length.

Taylor's got a long list of ex-lovers, and they'll tell you she's insane, but she's got a blank space baby, and she needs you to write your proof. Prove that Taylor Swift is correct: there is no string that will be colored with more than one color if every loop has an odd number of strings.

4. [12 pts] Real photograph of Taylor Swift in Towne!

Taylor Swift's favorite pasttime is proving facts about Markov's inequality on an balcony in summer air, as she knows it All Too Well. However, she was tripped up by the following proofs and needs you to help her!

(a) Show that Markov's inequality only applies to non-negative random variables. In other words, give an example of a random variable and its probability distribution for which Markov's inequality gives an incorrect answer.

(b) Suppose Z is a random variable that is always at least -6 and has expectation 0. Since Z can take on negative values, Markov's inequality does not apply directly. Still, show that the probability that $Z \geq 12$ is at most $\frac{1}{3}$. Use only Markov's inequality, and known properties about expectations and random variables.

(c) Now suppose we additionally know that Z has a variance of 12. With this information, show that $\Pr[Z \geq 12]$ is now at most $\frac{1}{12}$. Use only Markov's inequality, and known properties about expectations, variances and random variables.

5. [15 pts] BRIDGES

It's the start of their *Love Story* and Romeo is trying to make his way through the crowd towards Taylor as she stands there, on an balcony in summer air. He notices that each balcony at the venue is connected to every other balcony through a series of bidirectional bridges between balconies. Each bridge connects exactly two distinct balconies and any pair of balconies has at most one bridge between them.

Romeo wants to travel across all the bridges in an effort to find any potential way to get to Taylor. A big fan of Taylor but an even bigger fan of bridges, he wants to cross each bridge exactly once. Furthermore, he wants to end on a different balcony than the one he starts from. He doesn't necessarily have to start from the balcony he is on right now.

Romeo needs help proving that he is able to find such a route between balconies if and only if there are exactly two balconies which have an odd number of bridges to other balconies (i.e. all other balconies have an even number of bridges to other balconies). Will you help him satisfy his bridge desires? It's a graph question, baby, just say yes.