Mathematical Foundations of Computer Science
Solutions to Practice Problems for Exam 1
October 11, 2020

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1. For sets A, B, C, and D, suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$. Prove that if $x \notin D$ then $x \in B$.

Solution. We will prove the claim by proving the contrapositive. Suppose that $A \setminus B \subseteq C \cap D$ and $x \in A$ but $x \notin B$. Since $x \notin B$ and $x \in A$, it must be that $x \in A \setminus B$ and hence $x \in C \cap D$. Thus $x \in D$.

2. How many sequences of bits are there that have all of the following properties:

- Their length is either 5 or 7 or 9.
- Their middle bit is a 1.
- The number of 0's they have equals the number of 1's they have minus one.

(Give the answer and an explanation of how you obtained it. No proofs required.)

Solution. We will consider each length separately. For a length of m, where m is odd, the middle term is fixed to be 1. From the last condition, we know there are $\frac{m-1}{2}$ 0s in the sequence. Since the middle position is fixed, there are only $m-1$ bits we have control over. Thus, we can choose which of these $m-1$ bits will be our 0s. This gives a count of $\binom{m-1}{(m-1)/2}$ valid bit sequences when the length is m. We are looking for the sum of these values for when the length of the bit sequence (m) equals each of 5, 7, and 9. This gives a total count of $\binom{4}{2} + \binom{6}{3} + \binom{8}{4} = 96$ total valid bit sequences.

3. You are choosing a sequence of five characters for a license plate. Your choices for characters are any letter in PERM and any digit in 1223. Your five-character sequence can contain any of these characters at most the number of times they appear in either PERM or 1223. If there are no other restrictions, how many such sequences are possible?

Solution. Consider a partition on the set of possible plate sequences, in which we partition by the number of 2's that appear in the sequence. We can count the total number of sequences by determining the sum of the number of sequences in each case of the partition. There will be 3 cases, namely having 0, 1, or 2 2's in the sequence.

First, consider the case in which there are 0 2's in the sequence. There are $\binom{5}{0}$ ways to place the 0 2's into the 5 placeholders. This leaves 5 remaining placeholders to be filled in with the 6 remaining characters (which are P,E,R,M,1,3). Thus, there are $P(6,5)$ ways to do this. Therefore, the number of license plates in this case is $\binom{5}{0} * P(6,5)$.

Second, consider the case in which there is 1 2 in the sequence. There are $\binom{5}{1}$ ways to place the 1 2 into the 5 placeholders. This leaves 4 remaining placeholders to be filled in with the 6 remaining characters (which are P,E,R,M,1,3). Thus, there are $P(6,4)$ ways to do this. Therefore, the number of license plates in this case is $\binom{5}{1} * P(6,4)$.

Third, consider the case in which there are 2 2's in the sequence. There are $\binom{5}{2}$ ways to place the 2 2's into the 5 placeholders. This leaves 3 remaining placeholders to be filled in with the 6 remaining characters (which are P,E,R,M,1,3). Thus, there are $P(6,3)$ ways to do this. Therefore, the number of license plates in this case is $\binom{5}{2} * P(6,3)$.

Thus, the number of license plate sequences is:

$$\binom{5}{0} \times P(6,5) + \binom{5}{1} \times P(6,4) + \binom{5}{2} \times P(6,3) = 1 \times 720 + 5 \times 360 + 10 \times 120 = 3720$$

4. There are a variety of special hands that one can be dealt in poker. For each of the following types of hands, count the number of hands that have that type.
(a) Four of a kind: The hand contains four cards of the same numerical value (e.g., four jacks) and another card.
(b) Three of a kind: The hand contains three cards of the same numerical value and two other cards with two other numerical values.
(c) Flush: The hand contains five cards all of the same suit.
(d) Full house: The hand contains three cards of one value and two cards of another value.
(e) Straight: The five cards have consecutive numerical values, such as 7-8-9-10-jack. Treat ace as being higher than king but not less than 2. The suits are irrelevant.
(f) Straight flush: The hand is both a straight and a flush.

Solution.

(a) Four of a kind: There are 13 choices for the value of the four of a kind, and 48 other cards for the fifth card. $\binom{13}{1}\binom{48}{1} = 13 \cdot 48$.

(b) Three of a kind: 13 choices for a value, can choose 3 cards of that value, can choose 2 other cards of different values from the remaining 12 values, both of which can have any of four suits. $\binom{13}{1}\binom{4}{3}\binom{12}{2}\binom{4}{1}^2 = 13 \cdot 4 \cdot \binom{12}{2} \cdot 4^2$.

(c) Flush: Four different suits, and one can choose 5 possible values out of the 13 in that suit. $4 \cdot \binom{13}{5}$

(d) Full House: Need to choose 3 cards from one of the various values, and then 2 cards from another value. $13 \cdot \binom{4}{3} \cdot 12\binom{4}{2} = 13 \cdot 4 \cdot 12 \cdot 6$.

(e) Straight: A straight can start at any one of 9 values (cards 2–10) and each of the five cards can be of a different suit. $9 \cdot 4^5$

(f) Straight Flush: A straight flush can start at any of 9 different cards, and the straight can be of one of the fours suits. Hence, $9 \cdot 4$

5. Let A, B be arbitrary sets. Prove by contradiction that

$$A \subseteq B \implies A \setminus (A \cap B) = \emptyset.$$

You are NOT allowed to use in the proof *set algebra* facts (such as $A \subseteq B \iff A \cap B = A$ or $A \setminus A = \emptyset$). Your proof should use only the definitions of subset, set difference, intersection, and empty set and logical manipulation of statements.

Solution. Assume $A \subseteq B$. Suppose, for the sake of contradiction, that $A \setminus (A \cap B) \neq \emptyset$. Then there exists some $x \in A \setminus (A \cap B)$. By definition of set difference, $x \in A$ and $x \notin A \cap B$. The latter statement is equvialent to $\neg(x \in A \cap B) \equiv \neg(x \in A \wedge x \in B) \equiv \neg(x \in A) \vee \neg(x \in B) \equiv x \notin A \vee x \notin B$. However, we already determined that $x \in A$, so $x \notin B$. We assumed, though, that $A \subseteq B$, which means $x \in A \implies x \in B$. Thus, we have a contradiction, so our assumption that $A \setminus (A \cap B) \neq \emptyset$ must be false. Thus, $A \subseteq B \implies A \setminus (A \cap B) = \emptyset$.

6. Prove that if for some integer a, $a \geq 3$, then $a^2 > 2a + 1$.

Solution. First we assume that $a \geq 3$. We can note also that $3a > 2a + 1$ since $a > 1$. So if we can show that $a^2 \geq 3a$, then we're done. Note that $a^2 = a * a$, and $3a = 3 * a$. Since we know that $a \geq 3$, we can conclude $a * a \geq 3 * a$. Hence our proof is complete.

7. Give a combinatorial proof of the following identity for $N, a, b \in \mathbb{N}$:

$$\binom{N}{a}\binom{N}{b} = \sum_{i=0}^{\min(a,b)} \binom{N}{i}\binom{N-i}{a-i}\binom{N-a}{b-i}$$

Solution. Consider the situation: there are N people, and we will give a of them a boat, and b of them a house (some may receive both). We pose the question: how many ways can we do this? We can choose the people to give a boat to in $\binom{N}{a}$ ways, and we can choose the people to give a house to in $\binom{N}{b}$ ways. Thus there are $\binom{N}{a} * \binom{N}{b}$ total ways to distribute the boats and houses, which is exactly the left side of the equation.

The RHS starts with iterating on the number of people who are going to get both boats and houses (i iterates from 0 to $\min(a, b)$). We then choose the i people who receive both boats and houses. Next out of the remaining $N - i$ we choose $a - i$ who receive only boats. Note, that so far we have chosen $i + a - i = a$ people. Finally, out of the remaining $N - a$ people we choose $b - i$ who receive only houses.

8. Prove that $\sqrt{6}$ is irrational.

Solution. Assume for contradiction that $\sqrt{6}$ is rational. This implies that, for some relatively prime integers, m and n, where $n \neq 0$,

$$\begin{aligned} \sqrt{6} &= \frac{m}{n} \\ 6 &= \frac{m^2}{n^2} \\ 6n^2 &= m^2 \end{aligned} \tag{1}$$

We now consider the following cases.
Case 1: m is even and n is even. This is impossible, since m and n are relatively prime.
Case 2: m and n are both odd. We know from the lectures that the product of two odd numbers is odd. Thus, if m is odd then m^2 must be odd. This is impossible, since m^2 is a multiple of 6 (by equation (1)) and hence an even number.
Case 3: m is odd and n is even. This is also impossible by the same reasoning as Case 2.
Case 4: m is even and n is odd. Let $m = 2a$ and $n = 2b + 1$, for some integers a and b. Combining this with equation (1), we get

$$\begin{aligned} 6(2b+1)^2 &= (2a)^2 \\ 6(4b^2+4b+1) &= 4a^2 \\ 3(4b^2+4b+1) &= 2a^2 \end{aligned}$$

This is again impossible, as the left hand side of the above equation is a product of two odd numbers and hence is odd, but the right hand side is clearly an even number.

In each of the above cases we have arrived at a contradiction and hence we conclude that $\sqrt{6}$ is irrational.

9. There are 100 guests at a fundraising party, excluding the host. As part of a "fun" party game, the host pairs up the dinner guests into 50 pairs that the host calls "fundraising pairs". In the game, the individual with the smaller net worth in each pair declares the amount of money that they wish to donate, which the individual with the higher net worth must match in double. For example, if the individual with the smaller net worth in one pair donates \$100 dollars, the individual with the larger net worth must donate \$200 dollars.

The host says that the aim of the game is to raise a total of 9 million dollars between all of the individuals. Given this set up, how many ways can the game unfold? Assume that the net worth of each of the individuals is unique, that all donations are in whole dollars, and that all of them can donate up to 9 million dollars each.

Solution. First, let us calculate how many ways there are to pair up the individuals. We can do this by arranging all 100 individuals in line and interpreting the $2k+1$th and $2k+2$th individuals as paired up, where $0 \leq k \leq 49$, and then removing the ordering from these pairs. There are 100! ways of arranging the individuals in a line, which we divide by 50! to remove the ordering of the pairs, and by $(2!)^{50}$ to remove the ordering within each pair.

Second, we seek the number of possible ways that the individuals can donate money. Since the individual with the higher net worth must donate twice what the individual with the lower net worth donates, each pairs donation must be a multiple of 3. We can therefore find the number of arrangements using the stars and bars method seen in lecture. Each star is a donation of \$3 dollars, and there are 3 million stars. There are 49 bars since there are 50 pairs. Therefore, the number of arrangements is $\binom{3000000+49}{49}$.

Finally, we seek to the number of ways that the two individuals within the pair can donate the money. There is simply 1 way, since the ratio of donation between the two is fixed.

Therefore, the total number of ways that the game can unfold is $\frac{100!}{50!(2!)^{50}} \times \binom{3000049}{49}$.

10. Prove that for $n \in \mathbb{N}$, with $n \geq 2$, define s_n by

$$s_n = \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right).$$

Prove that $s_n = 1/n$ for every natural number $n \geq 2$.

Solution. Let $P(n)$ be the property that

$$\left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{n}\right) = \frac{1}{n}.$$

We will prove that $P(n)$ is true for $n \geq 2$ using induction on n.
Base Case: $P(2)$ is true because L.H.S. $= 1/2 =$ R.H.S.
Induction Hypothesis: Assume that $P(k)$ is true for some $k \geq 2$.
Induction Step: We want to prove that $P(k+1)$ is true. In other words, we want to prove that

$$\left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{k}\right) \times \left(1 - \frac{1}{k+1}\right) = \frac{1}{k+1}.$$

$$\begin{aligned}
\text{L.H.S.} &= \left(1 - \frac{1}{2}\right) \times \left(1 - \frac{1}{3}\right) \times \cdots \times \left(1 - \frac{1}{k}\right) \times \left(1 - \frac{1}{k+1}\right) \\
&= \frac{1}{k} \times \left(1 - \frac{1}{k+1}\right) \qquad \text{(using induction hypothesis)} \\
&= \frac{1}{k} \times \frac{k}{k+1} \\
&= \frac{1}{k+1}
\end{aligned}$$

11. Let n be a positive integer. Prove by induction on n that:

$$\sum_{\{a_1, a_2, \ldots, a_k\} \subseteq \{1, 2, \ldots, n\}} \frac{1}{a_1 a_2 \cdots a_k} = n$$

(Here the sum is over all non-empty subsets of $\{1, 2, \ldots, n\}$. For example, the set $\{1, 3, 6\}$ contributes $\frac{1}{1 \cdot 3 \cdot 6} = \frac{1}{18}$ to the sum.)

Solution. Let $P(n)$ be the predicate that the above equation holds for n. We will prove using induction on n that $P(n)$ is true for all $n \geq 1$.

Induction hypothesis: Assume that $P(j)$ is true for some $j \geq 1$.
Base Case: $P(1)$ is true because:

$$\sum_{\{a_1,a_2,\ldots,a_k\}\subseteq\{1\}} \frac{1}{a_1 a_2 \cdots a_k} = \frac{1}{1} = 1$$

Induction Step: We want to show that $P(j+1)$ is true, such that:

$$\sum_{\{a_1,a_2,\ldots,a_k\}\subseteq\{1,2,\ldots,j,j+1\}} \frac{1}{a_1 a_2 \cdots a_k} = j+1$$

The left hand side of this can be rewritten. We subdivide the summation's subsets into (a) subsets that do not include $j+1$, (b) subsets that include $j+1$ and at least one other element, and (c) the subset that is $\{j+1\}$:

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,j\}} \frac{1}{a_1 \cdots a_k} + \sum_{(\emptyset\neq\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,j\})\cup\{j+1\}} \frac{1}{a_1 \cdots a_k} + \sum_{\{a_1,\ldots,a_k\}\subseteq\{j+1\}} \frac{1}{a_1 \cdots a_k}$$

Notice that the terms of the second summation exactly match the terms of the first summation, but with an extra factor of $j+1$ in the denominator of each term. Thus the LHS of $P(j+1)$ reduces to:

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,j\}} \frac{1}{a_1 \cdots a_k} + \frac{1}{j+1} \times \sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,j\}} \frac{1}{a_1 \cdots a_k} + \frac{1}{j+1}$$

By the induction hypothesis, we know that:

$$\sum_{\{a_1,\ldots,a_k\}\subseteq\{1,\ldots,j\}} \frac{1}{a_1 \cdots a_k} = j$$

Substituting, we have:

$$j + \frac{j}{j+1} + \frac{1}{j+1} = j + \frac{j+1}{j+1} = j+1 = \text{RHS}$$

This we have shown that $P(j)$ implies $P(j+1)$, and this completes the proof.