2-SAT:

Input: Boolean formula $\phi_{n,m}$ in 2-CNF form.

n: # variables in ϕ

m: # clauses in ϕ

Output: TRUE, if ϕ is satisfiable.

FALSE, if ϕ is not satisfiable.

if TRUE then we want to obtain a truth assignment that will result in the formula ϕ evaluating to TRUE.

Example: $\phi : (x_1 \vee \bar{x}_2) \wedge (\bar{x}_2 \vee \bar{x}_3) \wedge (x_2 \vee x_4) \wedge (x_1 \vee \bar{x}_3)$

x_1, x_2, x_3, x_4 are variables.

$x_1, \bar{x}_1, x_2, \bar{x}_2, x_3, \bar{x}_3, x_4, \bar{x}_4$ are literals

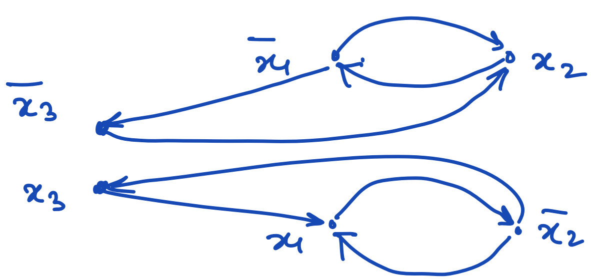
$(x_1 \vee \bar{x}_2)$ is a clause & so is $(x_2 \vee x_4)$. The above formula has four clauses.

We will solve the problem by reducing it to a graph problem. For any formula ϕ, we create an implication graph as follows:

- we have $2n$ vertices, one vertex corresponding to each literal.
- for each clause $(\alpha \vee \beta)$ we have two edges:
  - an edge $(\bar{\alpha}, \beta)$ and an edge $(\bar{\beta}, \alpha)$.

The intuition for creating edges this way is as follows: Consider a clause $(\alpha \vee \beta)$. Note that if α is not true then β must be true. That is, $\alpha \vee \beta \equiv \bar{\alpha} \Rightarrow \beta$. This is why we have the edge $(\bar{\alpha}, \beta)$. Similarly, for $(\bar{\beta}, \alpha)$.

<u>Example</u> : $\phi : (x_1 \vee x_2) \wedge (x_1 \vee \bar{x}_3) \wedge (x_2 \vee x_3) \wedge (\bar{x}_1 \vee \bar{x}_2)$



<u>Lemma</u>: ϕ is unsatisfiable iff $\exists i$ s.t.

- there is a path from x_i to $\bar{x}_i$ in G.

  AND

- there is a path from $\bar{x}_i$ to x_i in G.

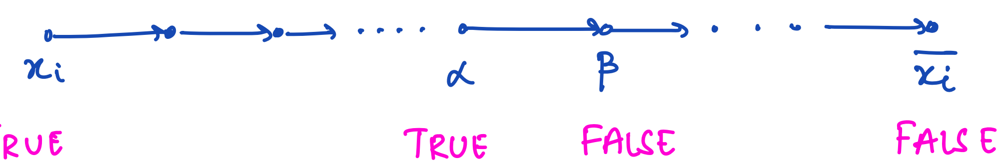
$(*)$

That is, ϕ is unsatisfiable iff $\exists i$ s.t. x_i & $\bar{x}_i$ are in the same SCC.

<u>Proof</u>: $(\Leftarrow)$ We will show that if $(*)$ holds then ϕ is unsatisfiable. Assume for contradiction that $(*)$ holds but ϕ is satisfiable. Let σ be a satisfying assignment of ϕ.

<u>Case I</u>: x_i is set to TRUE in σ.

Consider the path from x_i to $\bar{x}_i$ in G.



Since x_i is TRUE & $\bar{x}_i$ is FALSE, there must be an edge (α, β) on the path $x_i \rightsquigarrow \bar{x}_i$ s.t. α is TRUE & β is FALSE. Note that the clause corresponding to the edge (α, β) is $(\bar{\alpha} \vee \beta)$ which evaluates to FALSE since α is TRUE & β is FALSE. This contradicts that σ satisfies ϕ.

<u>Case II</u> : x_i is set to FALSE in ϕ.

Similar proof as above except that we now consider the path

$$\bar{x}_i \rightarrow \cdots \cdots \underset{\alpha}{\ } \longrightarrow \underset{\beta}{\ } \rightarrow \cdots \longrightarrow x_i \, .$$

We will prove the other direction by coming up with an algorithm.

1. Construct G

2. Compute G^{SCC}.

3. Compute a topological ordering of G^{SCC}.

4. Start with the sink vertex in G^{SCC}.

5. Set all literals in the sink vertex to be TRUE.

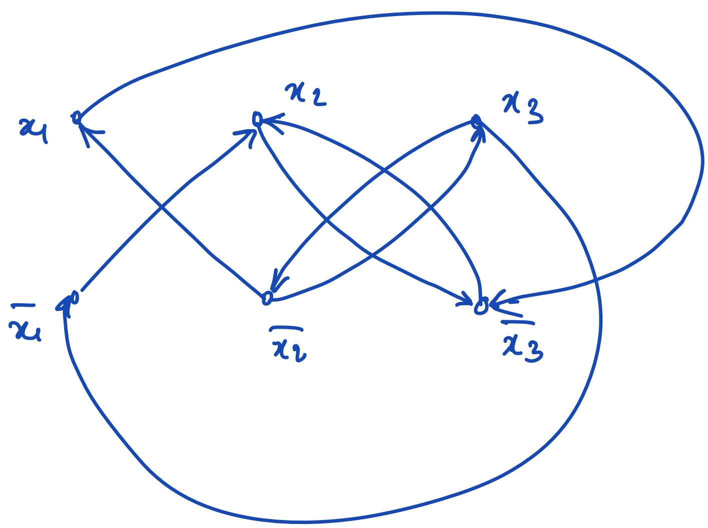
6. Set all the complement literals to False.
// All these literals must belong to the same
// SCC & this SCC must be a source in G^{SCC}.

7. Remove the sink & the source vertices from G^{SCC}.

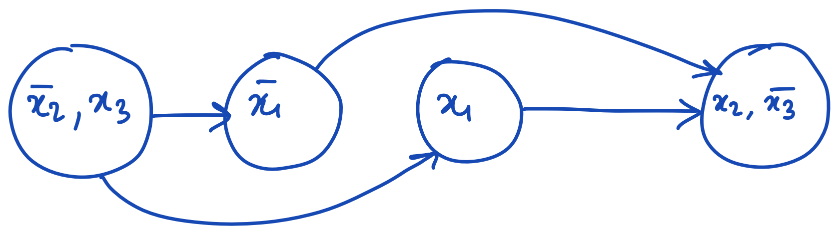
8. Repeat steps 4-7 until all variables are assigned a truth value.

Example

$\phi = (x_1 \vee x_2) \wedge (\overline{x}_1 \vee \overline{x}_3) \wedge (\overline{x}_2 \vee \overline{x}_3) \wedge (x_2 \vee x_3)$



G^{scc} :



1. We set x_2 to TRUE & $\overline{x}_3$ to TRUE.

2. We set $\overline{x}_2$ to FALSE & x_3 to FALSE.

3. Updated graph : $\overline{x}_1$ x_1

4. Set x_1 to TRUE & $\bar{x}_1$ to FALSE.

Thus our truth assignment is

x_1 = TRUE, x_2 = TRUE, x_3 = FALSE.

Note that there are other truth assignments as well.

Proof of Correctness

Lemma: If there is a path from α to β in G then there is a path from $\bar{\beta}$ to $\bar{\alpha}$ in G.

Proof: Consider a path from α to β in G

$$\alpha \to \alpha_1 \to \alpha_2 \to \alpha_3 \cdots \alpha_p \to \beta$$

The edges on this path imply that ϕ has

Clauses $(\bar{\alpha}_p \vee \beta), (\overline{\alpha_{p-1}} \vee \alpha_p), \ldots, (\bar{\alpha} \vee \alpha_1)$.

Thus we have edges

$(\bar{\beta}, \bar{\alpha}_p), (\bar{\alpha}_p, \bar{\alpha}_{p-1}), \ldots, (\bar{\alpha}_1, \bar{\alpha})$, i.e.,

we have the path

$$\bar{\beta} \to \bar{\alpha}_p \to \bar{\alpha}_{p-1} \to \cdots \cdots \bar{\alpha}_1 \to \bar{\alpha}$$

<u>Lemma</u>: Let ϕ be a 2-CNF formula with a satisfying truth assignment. Let α & β be literals belonging to the sink vertex of G^{SCC}. Then $\bar{\alpha}$ & $\bar{\beta}$ belong to a source vertex of G^{SCC}.

<u>Proof</u>: We will first prove that if α & β are in the same SCC in G then $\bar{\alpha}$ & $\bar{\beta}$ are also in the same SCC in G. By def^n, there is a path from $\alpha \rightsquigarrow \beta$ & a path from $\beta \rightsquigarrow \alpha$.

By the previous lemma, there must exist paths $\bar{\beta} \rightsquigarrow \bar{\alpha}$ and $\bar{\alpha} \rightsquigarrow \bar{\beta}$. This proves that $\bar{\alpha}$ & $\bar{\beta}$ must be in the same SCC.

It remains to show that $\bar{\alpha}$ & $\bar{\beta}$ are in a source vertex of G^{SCC}. Assume otherwise. If there is an edge from some vertex γ in a different SCC to $\bar{\alpha}$ then there must be an edge from α to $\bar{\gamma}$. Since γ & $\bar{\alpha}$ are in different SCCs, α & $\bar{\gamma}$ must also be in different SCCs. But this would mean that α does not belong to a sink node of G^{SCC}, a contradiction!

<u>Theorem</u>: If ϕ is satisfiable then our alg. correctly produces a valid truth assignment.

- Can be proved formally using the above lemmas & induction.